

MATH 23a, FALL 2002  
THEORETICAL LINEAR ALGEBRA  
AND MULTIVARIABLE CALCULUS  
Lectures # 1–3, supplement

Construction of the Integers,  $\mathbb{Z}$

Given the natural numbers,  $\mathbb{N}$ , and their arithmetic operations  $+$  and  $\cdot$ , we construct the integers as follows:

Let  $\mathbb{Z} = \{(a, b) | a, b \in \mathbb{N}\} / \sim$ , where  $(a, b) \sim (c, d)$  if and only if  $a + d = b + c$ .

Our rationale for doing so is to introduce additive inverses for the natural numbers by defining  $-1$  to be the equivalence class containing all pairs  $(a, b)$  whose “differences”  $b - a$  are  $-1$ , such as  $(1, 0)$ ,  $(2, 1)$ , etc. Note that according to this definition,  $0$  corresponds to the equivalence class of all pairs of the form  $(n, n)$ , and in general, the integer  $m$  corresponds to the equivalence class of all pairs of the form  $(n, n + m)$ .

Thus the elements of  $\mathbb{Z}$  are equivalence classes, and we will use the notation  $[(a, b)]$  to denote the equivalence class of the pair  $(a, b)$ . Given our set  $\mathbb{Z}$ , we must define the operations of addition and multiplication. We first consider addition. Thinking of the pairs by the differences they represent, adding  $(a, b)$  to  $(c, d)$  is the same as adding  $b - a$  to  $d - c$ , and so we want the result to be  $(b - a) + (d - c) = (b + d) - (a + c)$ . Thus, it makes sense to try defining:

Addition of two integers:  $[(a, b)] + [(c, d)] = [(a + c, b + d)]$

However, we should verify that this definition does not depend on the choice of representative from the equivalence class. That is, suppose  $(a', b') \in [(a, b)]$  and  $(c', d') \in [(c, d)]$ . Then we must show that the sum of these representatives is in the equivalence class  $[(a + c, b + d)]$ . We do this as follows:

If  $(a', b') \in [(a, b)]$ , then  $a' + b = a + b'$ , and similarly, if  $(c', d') \in [(c, d)]$ , then  $c' + d = c + d'$ . Hence, by adding the two left-hand members and the two right-hand members and re-arranging by associativity, we obtain:

$$(a' + c') + (b + d) = (a + c) + (b' + d')$$

According to the equivalence relation, another way to read this equality is to say that  $(a' + c', b' + d') \sim (a + c, b + d)$ , and thus the result of adding  $(a', b')$  to  $(c', d')$  is indeed in the correct equivalence class, and we say that the addition of two integers is now *well-defined*.