MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Lecture # 11, supplement

The Dimension of a Vector Space is Well-Defined

Reference: Halmos' "Finite-Dimensional Vector Spaces," Section 8

Theorem: Let V be a vector space over F. Any two bases for V have the same cardinality.

Remark: The general proof is outside the scope of this class and involves more serious set theory, including the Axiom of Choice.

Proof: We prove the result in the case that both bases have finite cardinality. Let $B_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $B_2 = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be two bases for V. We need to show that $m = n$, and we will do this by showing that $m \leq n$. By symmetry it will follow that $n \leq m$, and we will be done.

Consider the set
$$S = \{\mathbf{w}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

Since B_1 is a basis for V, the vectors in B_1 span V, and in particular, we may write $\mathbf{w}_1$ as a linear combination of the vectors in B_1. This implies that the set S is linearly dependent. (Recall our fact about writing one vector as a linear combination of the others.)

Now, starting with $\mathbf{w}_m$, construct larger and larger sets of linearly independent vectors from S. For example, we know that $\{\mathbf{w}_1\}$ is a linearly independent set because it contains only one non-zero vector. Now check the set $\{\mathbf{w}_m, \mathbf{v}_1\}$. If this is a linearly independent set, then add $\mathbf{v}_2$, and so on. At some point, however, we must have a linearly dependent set since we know that S itself is a linearly dependent set. *Take out the first vector $\mathbf{v}_i$ that makes the new set linearly dependent*, and make a new set

$$S' = \{\mathbf{w}_m, \mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\}.$$

We claim that S' spans V. Since S spans V, we know that we can write any vector $\mathbf{u} \in V$ as a linear combination of the elements of S. But $\mathbf{v}_i$ may be written as a linear combination of the vectors $\mathbf{w}_m, \mathbf{v}_1, \ldots, \mathbf{v}_{i-1}$ by the way we chose $\mathbf{v}_i$ above. Substitute this expression for $\mathbf{v}_i$ into the expression for $\mathbf{u}$, and we have written $\mathbf{u}$ as a linear combination of the vectors in S'.

Now repeat this process by adjoining $\mathbf{w}_{m-1}$ to S'. At each step, it must be one of the $\mathbf{v}$'s and not one of the $\mathbf{w}$'s that causes the set to become linearly dependent because we know that B_2 (the $\mathbf{w}$'s) is linearly independent.

Finally, we note that the $\mathbf{v}$'s cannot run out first! If they did, we would have one of the $\mathbf{w}$'s left over, which could then be written as a linear combination of the others since those others would span V.

Hence $m \leq n$.

Q.E.D.