MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Lecture # 24, supplement

Alternating n-linear forms on an n-dimensional space

Theorem: Let V be a vector space over F of dimension n. Then the vector space of alternating n-linear forms on V is one-dimensional over F.

Proof: We do this in two parts, first by showing that any two such alternating forms are scalar multiples of each other (so that the dimension of the space is at most one) and then by exhibiting a non-zero such form (so that the dimension is exactly one).

1. Let $f_1 : V^n \longrightarrow F$ and $f_2 : V^n \longrightarrow F$ be two alternating forms, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V.

   Let $\mathbf{w}_1, \ldots, \mathbf{w}_n \in V$ be any n vectors in V. Since the $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ form a basis, we may write each of the $\mathbf{w}$'s as a linear combination of the $\mathbf{v}$'s as follows:

   $$\mathbf{w}_i = \alpha_{1i}\mathbf{v}_1 + \cdots + \alpha_{ni}\mathbf{v}_n$$

   As in the previous theorem (that an alternating n-linear form applied to n linearly independent vectors in n-dimensional space is non-zero), we use the multilinearity of f_1 and f_2 to evaluate each of them:

   $$f_1(\mathbf{w}_1, \ldots, \mathbf{w}_n) = \sum_{\pi \in S_n} \alpha_\pi \cdot \mathrm{sgn}(\pi) \cdot f_1(\mathbf{v}_1, \ldots, \mathbf{v}_n)$$
   $$f_2(\mathbf{w}_1, \ldots, \mathbf{w}_n) = \sum_{\pi \in S_n} \alpha_\pi \cdot \mathrm{sgn}(\pi) \cdot f_2(\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

   where each α_π is a product of n of the coefficients α_{ij} from the expansions of the $\mathbf{w}$'s. The key observation is that these coefficients depend not on f_1 or f_2, but only on the coefficients of the $\mathbf{w}$'s!

   Lastly, since each of $f_1(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $f_2(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a single scalar, there clearly must exist two other non-zero scalars, say c_1 and c_2, such that

   $$c_1 \cdot f_1(\mathbf{v}_1, \ldots, \mathbf{v}_n) + c_2 \cdot f_2(\mathbf{v}_1, \ldots, \mathbf{v}_n) = 0$$

   and hence these same scalars serve to write

   $$c_1 \cdot f_1(\mathbf{w}_1, \ldots, \mathbf{w}_n) + c_2 \cdot f_2(\mathbf{w}_1, \ldots, \mathbf{w}_n) = 0,$$

   for *all* choices of the $\mathbf{w}$'s, and we see that f_1 and f_2 are linearly dependent.

2. There exists a non-zero alternating n-linear form on V.

   Choose the standard basis for V and write the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in coordinates with respect to this basis:

$$\mathbf{v}_1 = \begin{bmatrix} \alpha_{11} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n1} \end{bmatrix} \quad , \ldots , \quad \mathbf{v}_n = \begin{bmatrix} \alpha_{1n} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{nn} \end{bmatrix}$$

   Define $D : V^n \longrightarrow F$ by:

$$D(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \sum_{\pi \in S_n} (\mathrm{sgn}\pi) \cdot \alpha_{1\pi(1)} \cdots \alpha_{n\pi(n)}$$

   We leave it as an exercise to verify that D is in fact multilinear and alternating, though it is clear that D is non-zero because this definition immediately implies that $D(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.

Remark: We note that the definition of D in part 2 of the proof depends on a choice of coordinates. In fact, with a different choice of basis, the value of D on a particular set of vectors would most likely be different, though it will differ by a scalar multiple. However, because of part 1, all values of D will have changed by the same scalar!