

MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Answers to Quiz # 1
October 10, 2001

1. True or False

T or **F** If V is a vector space with $\dim(V) = n$, then any n vectors form a basis for V .

False. If the n vectors are linearly dependent, then they will not form a basis.

T or **F** If V is a finite-dimensional vector space, then any subspace is also finite-dimensional.

True. This is a theorem that we proved in class.

2. Let V be a vector space over a field F . Define what it means for the set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ to be *linearly independent*.

The vectors are *linearly independent* provided that:

if $a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n = \mathbf{0}$, then $a_1 = 0, \dots, a_n = 0$.

3. Consider the linear map $L : \mathbb{R}^3 \longrightarrow C[0, 1]$ defined by

$$L(a, b, c) = a \cdot 1 + b \cdot x + c \cdot (1 - x)$$

(a) Find the kernel K of L .

$Ker(L) = \{\mathbf{v} \in \mathbb{R}^3 | L(\mathbf{v}) = \mathbf{0}\}$. If $(a, b, c) \in \mathbb{R}^3$, then $L(a, b, c) = a \cdot 1 + b \cdot x + c \cdot (1 - x)$, which equals $(a + b) + (b - c) \cdot x$. Since this needs to equal to the zero vector in $C[0, 1]$, (namely, the zero function), for (a, b, c) to be in the kernel, both coefficients must be zero. Hence $a + b = 0$ and $b - c = 0$. But this means that we may choose c to be any real number, so long as we then follow by choosing $b = c$ and $a = -c$. In other words, the vector must look like $(-c, c, c)$. Hence $Ker(L) = \{(-c, c, c) | c \in \mathbb{R}\} = span\{(-1, 1, 1)\}$.

(b) What is the dimension of \mathbb{R}^3/K ?

Note that $\dim(\text{Ker}(L)) = 1$ since the kernel is spanned by one non-zero vector. Then we use the fact from the homework that the dimension of a quotient space is given by

$$\dim(U/V) = \dim(U) - \dim(V),$$

provided that these dimensions are finite, to see that $\dim(\mathbb{R}^3/K) = \dim(\mathbb{R}^3) - \dim(K) = 3 - 1 = 2$.

(c) Find a set of vectors (cosets) which spans \mathbb{R}^3/K .

Since $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ span \mathbb{R}^3 , their respective cosets $\mathbf{e}_1 + K$, $\mathbf{e}_2 + K$, and $\mathbf{e}_3 + K$ span \mathbb{R}^3/K . Note also that since $\dim(\mathbb{R}^3/K) = 2$, we should only require 2 vectors to span the quotient space, and in fact, any two of the above vectors (cosets) will suffice. If you selected two vectors (cosets) at random, not only should you check that no representative was in K but also that no linear combination of the representatives was in K .