MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm (take-home portion)
March 19, 2003

Directions: You have until class (11 A.M.) on Friday, April 4, to complete this exam, when it should be turned in to me or to my mailbox in the Science Center. You may use your own class notes, your own homework assignments, and the course textbooks (including the official course bibliography) as your only aids. You may not use any internet resources except for the course website and the posted homework solutions. You may not discuss the exam with anyone, and all questions should be directed only to the instructor. (In particular, please do not direct questions to the Math 23b CA's.)

There is one question worth ten points, and there is partial credit, but only for intelligible work. Please write neatly, and please turn in a clean copy of your solution, not scratch work that may or may not have anything to do with a final answer. One point will be awarded for *neatness only*, and one point will be awarded for *style only*. Make sure your name is prominently displayed on your work, and *please* staple your final pages together into one stack.

You may quote results from class and/or your notes with an appropriate reference, and you must cite anything you take from a published book. Otherwise, all work should be your own.

Definition.

A set $M \subset \mathbb{R}^m$ is said to be a **differentiable n-manifold** provided that there exists a set of pairs $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ called *charts* satisfying the following:

1. Each $U_\alpha \subset M$ and $M = \bigcup_{\alpha \in I} U_\alpha$.
2. Each φ_α is a map from U_α to $\mathbb{R}^n$ for which $E_\alpha = \mathrm{Im}(\varphi_\alpha)$ is an open subset of $\mathbb{R}^n$. Furthermore, $\varphi_\alpha : U_\alpha \longrightarrow E_\alpha$ is a *homeomorphism*, that is, a bijection such that both φ_α and φ_α^{-1} are continuous.
3. The charts are compatible in the sense that if (U, φ) and (V, ψ) are two charts with $U \cap V \neq \emptyset$, then the map

   $$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

   is a *diffeomorphism*, that is, a homeomorphism such that both the function and its inverse are in the class C^∞, or in other words, are infinitely differentiable.

Remarks:

(i) Note that n and m are fixed throughout and that $1 \leq n \leq m$. (The case $n = 0$ would consist of a collection of isolated points at best.)

(ii) Note that both $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$.

(iii) Though it is not a formal part of the definition, the U_α are generally taken to be connected.

(iv) The topology of M is given by the *relative topology* from $\mathbb{R}^m$, where a set $S \subset M$ is open in M if there exists some open set $T \subset \mathbb{R}^m$ such that $S = T \cap M$. In this sense, the U_α are open as subsets of M.

Problem: Choose one of the following:

A. Show that $S^1 \times S^1 = \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 = 1 \text{ and } z^2 + w^2 = 1\}$ is a differentiable 2-manifold.

B. Show that $S^3 = \{\mathbf{x} \in \mathbb{R}^4 \mid ||\mathbf{x}|| = 1\}$ is a differentiable 3-manifold.

For the manifold of your choosing, explicitly construct a set of charts and show that all maps satisfy the stated conditions.