MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm Solutions (in-class portion)
March 19, 2003

1. **True or False**

T or **F** Every bounded infinite set in $\mathbb{R}^n$ has an accumulation point.

True. This is the Bolzano-Weierstrass Theorem.

T or **F** Let $A \subset \mathbb{R}^n$. If $f : A \longrightarrow \mathbb{R}$ is continuous and f attains its maximum on A, then A is compact.

False. For example, $f(x) = 1 - x^2$ is continuous on all of $\mathbb{R}$, which is not compact, even though it does attain its maximum (of 1) at the point 0.

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous and $S \subset \mathbb{R}^n$ is connected, then $f(S) \subset \mathbb{R}^m$ is connected.

True. This is a theorem.

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$, then all of its directional derivatives exist at $\mathbf{a} \in \mathbb{R}^n$.

True. This is a theorem.

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $\mathbf{a}$, and $\mathbf{h}$ is some direction vector, then $D_{\mathbf{h}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{h}$.

True. This is a theorem.

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$, then $[D_i D_j f](\mathbf{a}) = [D_j D_i f](\mathbf{a})$, for all i and j.

False. There are differentiable functions whose second-order partial derivatives do not even exist.

T or **F** On the set of points $S = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$, where $f(x, y) = x^2 - y^2 - 1$, there is a neighborhood of the point $(1, 0)$ on which we may write $y = h(x)$ with $f(x, h(x)) = 0$.

False. First note that the Implicit Function Theorem does not apply because $\frac{\partial f}{\partial y} = 2y$ and that this is zero at the point $(1, 0)$. Of course, this does not guarantee that y cannot be written as a function of x, but an examination of the graph of points where $f(x, y) = 0$ shows that in any neighborhood of this point, there are points of the form $(x_0, \pm\sqrt{x_0^2 - 1})$, so that y cannot be uniquely determined as a function of x.

T or **F** Let $A \in M_n(\mathbb{R})$. Then A is symmetric if and only if A has an orthonormal eigenbasis.

True. This is the Spectral Theorem for Real Symmetric Matrices.

T or **F** The quadratic form $q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $q(x, y) = x^2 + 4xy + y^2$ is positive-definite.

False. Note that $q(1, 1) = 6 > 0 > -2 = q(1, -1)$, or note that the eigenvalues of the associated matrix $A = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$ are 3 and -1.

2. Compactness. (12 points, 3/3/6)

(a) Define what it means for a set $A \subset \mathbb{R}^n$ to be compact.

A set $A \subset \mathbb{R}^n$ is compact if any open cover of A has a finite subcover. In other words, if $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets such that $A \subset \bigcup_{\alpha \in I} U_\alpha$, then there is a finite subset $J \subset I$ such that $A \subset \bigcup_{\alpha \in J} U_\alpha$.

(b) State the Heine-Borel Theorem.

A set $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded. (Technically, it would also be correct to say that the Heine-Borel Theorem is the "if" part of the above statement.)

(c) Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous and $A \subset \mathbb{R}^n$ is compact. Show that $f(A) \subset \mathbb{R}^m$ is compact.

Let $\{V_\alpha\}_{\alpha \in I}$ be any collection of open sets that covers $f(A)$. Since f is continuous, each $f^{-1}(V_\alpha)$ is open as a subset of $\mathbb{R}^n$, and the collection $\{f^{-1}(V_\alpha)\}_{\alpha \in I}$ covers A by the definition of $f(A)$. But since A is compact, there is a finite subcollection $\{f^{-1}(V_\alpha)\}_{\alpha \in J}$ that also covers A, and then the collection $\{V_\alpha\}_{\alpha \in J}$ covers $f(A)$, so $f(A)$ is compact.

3. Differentiability. (12 points, 3/3/6)

(a) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.
Define what it means for f to be *differentiable* at $\mathbf{a} \in \mathbb{R}^n$.

The function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$ provided that there exists a linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{||\mathbf{h}|| \to 0} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})}{||\mathbf{h}||} = \mathbf{0}.$$

(b) Give an example of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ that is continuous at all $\mathbf{x} \in \mathbb{R}^n$, but which is not differentiable at some point $\mathbf{a} \in \mathbb{R}^n$.

(You should specify a, but you need not prove the continuity or the lack of differentiability of f, as long as the example is correct.)

The easiest example is $f(\mathbf{x}) = ||\mathbf{x}||$, which is continuous everywhere but not differentiable at $\mathbf{0}$. (In fact, none of the partials exist at $\mathbf{0}$.)

(c) Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is bounded (in other words, $\exists M > 0$ such that $|f(\mathbf{x})| \leq M, \forall \mathbf{x} \in \mathbb{R}^n$), and define $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$g(x, y) = x \cdot y \cdot f(x, y).$$

Show that g is differentiable at $(0, 0)$.

Since g is behaving like the function $h(x, y) = cxy$ near $(0, 0)$, we guess that $\nabla g(0, 0) = \nabla h(0, 0) = (0, 0)$.
We evaluate the limit from the definition of differentiability as follows (using absolute values since we anticipate the limit being 0), using $\mathbf{h} = (h, k)$ from the second step onward:

$$\begin{aligned}
\lim_{||\mathbf{h}|| \to 0} \left| \frac{g(\mathbf{0} + \mathbf{h}) - g(\mathbf{0}) - L(\mathbf{0})}{||\mathbf{h}||} \right| &= \lim_{||\mathbf{h}|| \to 0} \frac{|g(\mathbf{h})|}{||\mathbf{h}||} \\
&= \lim_{||\mathbf{h}|| \to 0} \frac{|h \cdot k \cdot f(h, k)|}{||\mathbf{h}||} \\
&\leq M \cdot \lim_{||\mathbf{h}|| \to 0} \frac{|h \cdot k|}{||\mathbf{h}||} \\
&\leq M \cdot \lim_{||\mathbf{h}|| \to 0} \frac{||\mathbf{h}||^2}{||\mathbf{h}||} \\
&= 0
\end{aligned}$$

where we have used the fact that f is bounded in the third step and the Cauchy-Schwarz inequality in the next-to-last step.

4. Inverse Function Theorem. (6 points, 3 each)

Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ **given by:**

$$f(x, y) = (e^{xy}, e^{x+y})$$

(a) Find the Jacobian, Jf**.**

$$Jf(x, y) = \begin{bmatrix} y \cdot e^{xy} & x \cdot e^{xy} \\ e^{x+y} & e^{x+y} \end{bmatrix}$$

(b) Find all points $(x, y) \in \mathbb{R}^2$ **at which the Inverse Function Theorem does not apply to** f**.**

To apply the Inverse Function Theorem, we compute:

$$\begin{aligned} \det Jf(x, y) &= y \cdot e^{xy} \cdot e^{x+y} - x \cdot e^{xy} \cdot e^{x+y} \\ &= (y - x) \cdot e^{xy+x+y}. \end{aligned}$$

Since $e^a > 0$ for any $a \in \mathbb{R}$, we have $\det Jf = 0$ precisely when $x = y$, so these are the points at which the Inverse Function Theorem does not apply.

5. Optimization of Functions. (6 points)

Use critical point classification and the method of Lagrange multipliers to find the point(s) on the closed unit sphere

$$D^3 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}$$

(which includes its interior) where the function

$$f(x, y, z) = x^3 + y^3 + z^3$$

attains its maximum and minimum.

First note that D^3 is closed and bounded and hence compact. Since f is continuous, it attains its max and min somewhere on D^3.

The function f is also differentiable, and $\nabla f(x, y, z) = (3x^2, 3y^2, 3z^2)$. Thus, the only critical point in the interior of D^3 occurs when $\nabla f = \mathbf{0}$, namely at the point $\mathbf{0} \in D^3$.

As for $\partial D^3 = S^2$, we parametrize it as the zero set of the function $g(x, y, z) = x^2 + y^2 + z^2 - 1$. For this g, we have $\nabla g(x, y, z) = (2x, 2y, 2z)$. The method of Lagrange multipliers tells us that the only possible max/min of f on the boundary occurs if there is some $\lambda \in \mathbb{R}$ such that:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z).$$

This yields the system of equations (including the constraint defined by $g = \mathbf{0}$):

$$3x^2 = 2\lambda x$$
$$3y^2 = 2\lambda y$$
$$3z^2 = 2\lambda z$$
$$x^2 + y^2 + z^2 - 1 = 0$$

There are 14 solutions to this set of equations:

$$\mathbf{v}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \mathbf{v}_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \mathbf{v}_5 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \mathbf{v}_7 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right),$$

$$\mathbf{v}_9 = (1, 0, 0)\,, \mathbf{v}_{11} = (0, 1, 0)\,, \mathbf{v}_{13} = (0, 0, 1)\,, \text{ and } \mathbf{v}_{2n} = -\mathbf{v}_{2n-1} \text{ for } n = 1, \ldots, 7.$$

At these fifteen points (including $\mathbf{0}$), f attains a maximum of 1 at $\mathbf{v}_9$, $\mathbf{v}_{11}$, and $\mathbf{v}_{13}$ and a minimum of -1 at $\mathbf{v}_{10}$,$\mathbf{v}_{12}$, and $\mathbf{v}_{14}$.