MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Questions for the Final

Directions: Here are four questions to help you prepare for the final, which will take place at 9:15 A.M. on Friday, May 16 in Lecture Hall A of the Science Center. At least one of these questions will be on the final, either verbatim or in some small variation.

You are encouraged to work with other members of the class on these problems or to discuss them with the instructor, but in the interests of fairness, please do not ask the course assistants about these problems.

1. Let $f : [a, b] \to \mathbb{R}$ be an *increasing* function, that is, if $x, y \in [a, b]$ and $x < y$, then $f(x) < f(y)$. Show that f is integrable on $[a, b]$.

2. Let $q : \mathbb{R}^n \to \mathbb{R}$ be a positive-definite, symmetric quadratic form, and let $X = \{\mathbf{x} \in \mathbb{R}^n \mid q(\mathbf{x}) \leq 1\}$. Compute $v(X)$, the n-volume of X.

3. Let $A \subset \mathbb{R}^n$ be a closed rectangle, and consider the vector space

$$I = \{f : A \longrightarrow \mathbb{R} \mid f \text{ is integrable on } A\}.$$

We define a map $|| \cdot ||_1 : I \longrightarrow \mathbb{R}$ by $||f||_1 = \int_A |f|.$

The map $||\cdot||_1$ satisfies the rules for being a norm on I except that there exist functions $f \neq 0$ such that $||f||_1 = 0$ (example?); in other words, this map is positive but not positive-definite (we call such a map a *semi-norm*). There are two possible modifications we can make:

(a) Let $J = \{f \in I \mid f \text{ is continuous on } A\}$ be a subspace of I. Show that $|| \cdot ||_1$ is positive definite on J by showing that if $f \neq 0$, then $||f||_1 > 0$.

(b) Let $K = \{f \in I \mid \int_A |f| = 0\}$ be a subspace of I. If we define the quotient space I/K, then two functions $f, g \in I$ are equivalent if $f - g \in K$, that is, if $\int_A |f - g| = 0$. Show that $|| \cdot ||_1$ defined on I/K by $||f + K||_1 = \int_A |f|$ is well-defined and positive definite.

4. **Stokes' Theorem.**

Let S be a parametrized surface in $\mathbb{R}^3$. That is, there exists some $D \subset \mathbb{R}^2$ that is open, connected, and simply-connected, with a boundary $C = \partial D$ that is a piece-wise smooth, positively-oriented curve, and a bijective function $\varphi : \overline{D} \to S$ that is continuous on $\overline{D}$ and continuously differentiable (except possibly on a set of measure zero) on D.

Let $F : U \to \mathbb{R}^3$ be a vector field that is class C^1 and represented by the components $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$, where $U \subset \mathbb{R}^3$ is some open set containing $\overline{D}$.

Then Stokes' Theorem says that:

$$\int\int_S \operatorname{curl} F = \int_C F$$

where, if

$$G = \operatorname{curl} F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right),$$

then with the parametrization above,

$$\int\int_S \operatorname{curl} F = \int\int_S G \cdot \mathbf{n} \, dS$$

$$= \int\int_D G(\varphi(u, v)) \cdot (\varphi_u \times \varphi_v) \, du \, dv,$$

where $\mathbf{n}$ is the normal vector to the surface S, and u and v are the two variables parametrizing D as a subset of $\mathbb{R}^2$. (For more, please read Chapter 19.3 in Fitzpatrick.)

(a) Prove Stokes' Theorem in the case that S is a rectangle parallel to the yz-plane (and whose edges are parallel to the y and z axes), that is, one of the form

$$S = \{(x, y, z) \mid x = a, b \le y \le b', c \le z \le c'\},$$

for some constants $a, b, c \in \mathbb{R}$.

(b) Verify Stokes' Theorem for the vector field $F(x, y, z) = (z, x, y)$ on the surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}$, the upper-hemisphere of the sphere of radius one, centered at the origin.