MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Solutions to Final Exam (take-home portion)

1. The group $GL_n(\mathbb{R})$

Consider the vector space $V = \mathbb{R}^n$. We define the collection of linear transformations (and their matrices with respect to the standard basis) from V to V to be:

$$M_n(\mathbb{R}) = \{A : V \longrightarrow V |\ A \textbf{ is linear}\}.$$

We further define the collection (which has the algebraic structure of a *group*) of invertible linear transformations, called the *general linear group*, to be:

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) |\ A \textbf{ is invertible}\}.$$

(a) Show that the determinant function,

$$\mathbf{det} : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$$

is continuous.

If $A = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \in M_n(\mathbb{R})$, then $det(A) = \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \cdot \alpha_{1\pi(1)} \cdots \alpha_{n\pi(n)}$.

We proved in class that any coordinate projection such as $f : \mathbb{R}^k \to \mathbb{R}$ with $f(x_1, \ldots, x_k) = x_i$ for some fixed i with $1 \leq i \leq k$ is continuous. We also proved that the product, sum, and scalar multiple of any continuous functions are continuous. Note that the determinant is just such a sum of products of coordinate projections (some multiplied by -1), and so it is continuous.

(b) Show that $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$.

Consider the set $U = \mathbb{R} \setminus \{0\}$. This set is open in $\mathbb{R}$ since for any $x \in U$, if we choose $\varepsilon = |x|$, we have $B_\varepsilon(x) \subset U$.

Now we note that the matrix A is in $GL_n(\mathbb{R})$ if and only if $det(A) \neq 0$. In other words, $GL_n(\mathbb{R}) = (det)^{-1}(U)$. Since U is open and det is continuous (from part (a)), it follows from our theorem about continuity that $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$.

(c) **Show that the closure of $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ is all of $M_n(\mathbb{R})$.**

Let $A \in M_n(\mathbb{R})$, and consider the sequence of matrices $\{A_k\}_{k=1}^{\infty}$, where $A_k = A - \frac{1}{k}I$. It is easy to see that this sequence converges to A because, given any $\varepsilon > 0$, we choose $N > \frac{\sqrt{n}}{\varepsilon}$. If $k > N$, then $||A_k - A|| = \sqrt{n \cdot \frac{1}{k^2}} < \varepsilon$.

Now we consider the question of whether these matrices A_k are invertible. Suppose A_k is not invertible. Then $det(A_k) = det(A - \frac{1}{k}I) = 0$. This implies that $\lambda = \frac{1}{k}$ is an eigenvalue for A. Since A is $n \times n$, we know that it has at most n distinct eigenvalues, and so there at most finitely many k such that $\frac{1}{k}$ is an eigenvalue for A. In other words, there at most finitely many k for which A_k is not invertible.

Let N be the largest such k. Then the sequence $\{A_k\}_{k=N+1}^{\infty}$ is a sequence of invertible matrices converging to A, and so A is a limit point of $GL_n(\mathbb{R})$.

2. Nilpotent Matrices

Let $V = \mathbb{R}^n$, and let $A : V \to V$. We say that A is *nilpotent* if there is some $m \in \mathbb{N}$ such that $A^m = 0$, and we say that m is the *degree* of nilpotency if $A^m = 0$ but $A^{(m-1)} \neq 0$.

(a) **Let $A : V \to V$ be nilpotent of degree m. Let $\mathbf{v} \in V$ be such that $A^{(m-1)}\mathbf{v} \neq \mathbf{0}$. Show that $\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \ldots, A^{(m-1)}\mathbf{v}\}$ is a linearly independent set in V. Conclude that $m \leq n$.**

Suppose $c_0\mathbf{v} + c_1 A\mathbf{v} + c_2 A^2\mathbf{v} + \cdots + c_{m-1}A^{(m-1)}\mathbf{v} = \mathbf{0}$. If we apply the linear map A^{m-1} to both sides and use linearity to factor out scalars, we get the equation

$$c_0 A^{(m-1)}\mathbf{v} + c_1 A^m \mathbf{v} + c_2 A^{(m+1)}\mathbf{v} + \cdots + c_{m-1}A^{(2m-2)}\mathbf{v} = \mathbf{0}.$$

Since A is nilpotent of degree m, this simplifies to $c_0 A^{(m-1)}\mathbf{v} = \mathbf{0}$, and since we know by hypothesis that $A^{(m-1)}\mathbf{v} \neq \mathbf{0}$, we conclude that $c_0 = 0$.

Repeating this process with $A^{(m-2)}$, we conclude that $c_1 = 0$, and by repeating this or using mathematical induction, we may further conclude that $c_i = 0$ for each $1 \leq i \leq m-1$, and hence these m vectors are linearly dependent.

As we have proven in class, any collection of linearly independent vectors in a vector space of dimension n must not be greater than n in number, and hence we conclude that $m \leq n$.

(b) **Exhibit examples of nilpotent matrices of degrees 1,2, and 3.**

Note that the original question asked for a matrix that was nilpotent of degree 0, but this is impossible since, by definition, $A^0 = I$, for any matrix $A \in M_n(\mathbb{R})$.

For the other possible degrees, we take matrices from $M_3(\mathbb{R})$.

Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

It is a straightforward exercise in matrix multiplication to see that A, B, and C have degrees 1, 2, and 3, respectively.

(c) **Let $A : V \to V$ be nilpotent. Find all the eigenvalues of A.**

Suppose λ is an eigenvalue of A, and let $\mathbf{v}$ be a non-zero eigenvector with this eigenvalue. Then $A\mathbf{v} = \lambda\mathbf{v}$, and by linearity, we have $A^m\mathbf{v} = \lambda^m\mathbf{v}$. Since A is nilpotent of degree m, this means that $\lambda^m\mathbf{v} = \mathbf{0}$. The only way this may occur is if $\lambda = 0$, and thus we have discovered the only *possible* eigenvalue for A.

To see that this eigenvalue in fact occurs, consider a vector $\mathbf{v}$ such that $A^{(m-1)}\mathbf{v} \neq \mathbf{0}$. Such a vector exists from the definition of the degree of nilpotency. Then $A(A^{(m-1)}\mathbf{v}) = A^m\mathbf{v} = \mathbf{0}$, and hence $A^{(m-1)}\mathbf{v}$ is a non-zero vector with eigenvalue 0.

(d) **Show that if A is nilpotent, then $I + A$ is invertible. (If A is nilpotent, then $I + A$ is called *unipotent*.)**

Suppose $I + A$ is not invertible. Then $0 = det(I + A) = det(A - (-1)I)$, and hence -1 is a root of the characteristic polynomial of A. But this would imply that -1 is an eigenvalue of A, which we know not to be the case from part (c).

3. “Space-Time”

Let $V = \mathbb{R}^3$, and consider the bilinear map $f : V \times V \to \mathbb{R}$ given by

$$f(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^t B \mathbf{v}, \quad \textbf{where} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This form f defines a new geometry on V, which has orthogonality defined by $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ and length defined by $||\mathbf{u}|| = \sqrt{|\langle \mathbf{u}, \mathbf{u} \rangle|}$.

This geometry differs from Euclidean geometry because we find that there are points/vectors whose behaviors vary according to f:

- **If $\langle \mathbf{u}, \mathbf{u} \rangle > 0$, we say that u is space-like.**
- **If $\langle \mathbf{u}, \mathbf{u} \rangle = 0$, we say that u is light-like.**
- **If $\langle \mathbf{u}, \mathbf{u} \rangle < 0$, we say that u is time-like.**

Because we still have length and orthogonality, we may still speak of an “orthonormal basis” and the “orthogonal complement” of a subspace $W \subset V$, but now defined in terms of f.

(a) Show that f is not an inner product.

(b) Show that f is not an alternating form.

It is possible to answer parts (a) and (b) together. For f to be an inner-product, we would need to have (among other things) $f(\mathbf{v}, \mathbf{v}) \geq 0, \forall \mathbf{v} \in \mathbb{R}^3$, and for f to be an alternating form, we would need to have (again, among other things) $f(\mathbf{v}, \mathbf{v}) = 0, \forall \mathbf{v} \in \mathbb{R}^3$.

Observe that for the vector $\mathbf{e}_3 = (0, 0, 1)$, we have $f(\mathbf{e}_3, \mathbf{e}_3) = -1$, and hence f is neither an inner-product nor an alternating form.

(c) Show that any orthonormal basis for V must consist of two space-like vectors and one time-like vector.

Suppose $\mathfrak{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an orthonormal basis for $\mathbb{R}^3$ with respect to the bilinear form f, with $\mathbf{u} = (a, b, c)$, $\mathbf{v} = (i, j, k)$, and $\mathbf{w} = (x, y, z)$.

We show the result in three steps:

i. $\mathfrak{B}$ contains no light-like vectors. Suppose $\mathbf{u}$ were light-like. Then $f(\mathbf{u},\mathbf{u}) = a^2+b^2-c^2 = 0$ which means that $||\mathbf{u}|| = 0$, and hence $\mathbf{u}$ is not part of an orthonormal basis.

ii. $\mathfrak{B}$ contains at most one time-like vector. Suppose $\mathbf{v}$ and $\mathbf{w}$ were both time-like. Then $f(\mathbf{v},\mathbf{v}) = i^2+j^2-k^2 < 0$ and $f(\mathbf{w},\mathbf{w}) = x^2+y^2-z^2 < 0$, and since they are supposed to be orthogonal, we also have $f(\mathbf{v},\mathbf{w}) = ix+jy-kz = 0$. Manipulating these inequalities and equations yields the following:

$$i^2+j^2 < k^2 \text{ and } x^2+y^2 < z^2 \text{ and hence } (i^2+j^2)(x^2+y^2) < k^2z^2$$

Multiplying out and manipulating further:

$$(i^2x^2+2ijxy+j^2y^2)+(i^2y^2-2ijxy+j^2x^2) < k^2z^2$$

$$(ix+jy)^2+(iy-jx)^2 < k^2z^2$$

But since $ix+jy = kz$, this simplifies to $(iy-jx)^2 < 0$, which is clearly impossible.

iii. $\mathfrak{B}$ cannot contain three-space like vectors. Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ were all space-like. Then

$$a^2+b^2 > c^2 \text{ and } i^2+j^2 > k^2 \text{ and } x^2+y^2 > z^2$$

and by orthogonality,

$$ai+bj = ck \text{ and } ax+by = cz \text{ and } ix+jy = kz.$$

Now suppose $\mathbf{s} = p\mathbf{u}+q\mathbf{v}+r\mathbf{w} = (pa+qi+rx, pb+qj+ry, pc+qk+rz)$ is any other vector in $\mathbb{R}^3$. Then

$$\begin{aligned} f(\mathbf{s},\mathbf{s}) &= (pa+qi+rx)^2+(pb+qj+ry)^2-(pc+qk+rz)^2 \\ &= p^2(a^2+b^2-c^2)+q^2(i^2+j^2-k^2)+r^2(x^2+y^2-z^2) \\ &\quad +2pq(ai+bj-cz)+2pr(ax+by-cz)+2qr(ix+jy-kz) \end{aligned}$$

By the orthogonality condition, the last three of these terms are 0. By inspection, we see that the first three terms are all positive, and hence $f(\mathbf{s},\mathbf{s}) \geq 0$ for any $\mathbf{s}$ which is a linear combination of the vectors in $\mathfrak{B}$, that is, for any vector in $\mathbb{R}^3$. In other words, this would imply that *every* vector is space-like, but we already know that there exist time-like vectors such as $\mathbf{e}_3 = (0,0,1)$, so we have our contradiction.

Putting all of this together, any orthonormal basis has no light-like vectors, and at most one time-like vector. Since not all three can be space-like, we must have two space-like and one time-like.

And just for the record, there do exist orthonormal bases with respect to f. For example, the standard basis is one.

(d) Let $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$**, and let** $W = \textbf{span}\{\mathbf{w}\}$**. Find a basis for** $W^{\perp}$ **and show that** $W \cap W^{\perp} \neq \{\mathbf{0}\}$**.**

Recall that $W^{\perp} = \{\mathbf{v} \in \mathbb{R}^3 | f(\mathbf{v}, \mathbf{w}) = 0\}$. If $\mathbf{v} = (a, b, c) \in W^{\perp}$, this translates to the condition $3a + 4b - 5c = 0$.

One basis for $W^{\perp}$ consists of the vectors $\mathbf{v}_1 = (5, 0, 3)$ and $\mathbf{v}_2 = (0, 5, 4)$. To show that these are linearly independent, if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$, then $(5c_1, 4c_2, 3c_1 + 4c_2) = (0, 0, 0)$, which clearly can happen only when $c_1 = c_2 = 0$. To show that they span, if we have any vector (a, b, c) satisfying $3a + 4b - 5c = 0$, we may write it as $(a, b, c) = \frac{a}{5}(5, 0, 3) + \frac{b}{4}(0, 4, 3)$.

Finally, we note that the vector $\mathbf{w}$ itself is in $W^{\perp}$ since $f(\mathbf{w}, \mathbf{w}) = 0$, and hence $W \cap W^{\perp} \neq \{\mathbf{0}\}$.

4. Upper Triangular Matrices and an Application to Differential Equations

Consider $V = \mathbb{R}^2$, and as in problem #1, define

$$M_2(\mathbb{R}) = \{A : V \longrightarrow V |\ A \textbf{ is linear}\}.$$

We have seen that if $A \in M_2(\mathbb{R})$ has two distinct real eigenvalues, say λ_1 and λ_2, then A is diagonalizable, and there is some basis for V with respect to which A may be diagonalized and written in the form $S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, for some invertible matrix S. We have also seen examples of matrices in $M_2(\mathbb{R})$, such as rotations like $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, that have no real eigenvalues.

(a) **Show that if $A \in M_2(\mathbb{R})$ has exactly one real eigenvalue, say λ_0, then there is some basis for V with respect to which A may be diagonalized and written in the form $S^{-1}AS = \begin{bmatrix} \lambda_0 & b \\ 0 & \lambda_0 \end{bmatrix}$.**

Suppose $A \in M_2(\mathbb{R})$ has one real eigenvalue, λ_0. Since λ_0 is an eigenvalue, there is some non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda_0\mathbf{v}$. Now choose any other non-zero vector $\mathbf{u}$ which is not a scalar multiple of $\mathbf{v}$ (so that $\mathbf{v}$ and $\mathbf{u}$ are linearly independent). With respect to the basis $\mathfrak{B} = \{\mathbf{v}, \mathbf{u}\}$, what is the matrix of A?

Note that $A\mathbf{v} = \lambda_0\mathbf{v} + 0\mathbf{u}$, and so the first column of A with respect to $\mathfrak{B}$ must be $\left[\begin{smallmatrix} \lambda_0 \\ 0 \end{smallmatrix}\right]$.

On the other hand, $A\mathbf{u} = b\mathbf{v} + d\mathbf{u}$, for some scalars $b, d \in \mathbb{R}$ since $\mathfrak{B}$ is a basis for $\mathbb{R}^2$, and so the second column of A is $\left[\begin{smallmatrix} b \\ d \end{smallmatrix}\right]$.

Thus far, we know that $A = \left[\begin{smallmatrix} \lambda_0 & b \\ 0 & d \end{smallmatrix}\right]$, but if this is so, then the characteristic polynomial of A would be $f_A(\lambda) = (\lambda - \lambda_0)(\lambda - d)$, which matches what we would expect from an upper-triangular matrix. However, this would imply that d was also an eigenvalue of A, and since we know by hypothesis that λ_0 is the only eigenvalue, we must conclude that $d = \lambda_0$.

(b) **(Not required) Observe that any such matrix A (from**

part (a)) may be decomposed as follows:

$$\underbrace{\begin{bmatrix} \lambda_0 & b \\ 0 & \lambda_0 \end{bmatrix}}_{S^{-1}AS} = \underbrace{\begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}}_{N},$$

where D is a diagonal matrix, and N is a nilpotent matrix (see problem #2). Furthermore, note that $DN = ND$.

Now we consider an application to linear differential equations. Consider the second-order differential equation:

$$y'' + ay' + by = 0, \quad \textbf{where } a, b \in \mathbb{R}.$$

A solution to this equation is a function y, considered to have the single variable t. This equation is called an *ordinary* differential equation because there is only one independent variable for the function y. It is *second-order* because it involves the second-derivative of y and no higher order derivatives. It is said to be *homogeneous* because every term involves y or one of its derivatives.

We use linear algebra to solve this equation in the following manner. (See Curtis, Section 34, for more information, which we only summarize here!) Let $v_1 = y$ and let $v_2 = y'$. Then we can express this differential equation as a system of linear differential equations:

$$\frac{dv_1}{dt} = v_2$$

$$\frac{dv_2}{dt} = -av_2 - bv_1$$

Note that we can write this system as a matrix equation:

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

or in other words, $\frac{d\mathbf{v}}{dt} = A\mathbf{v}$, where $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

Surprisingly, this equation has the solution $\mathbf{v}(t) = \mathbf{v}_0 \cdot e^{At}$ (where $\mathbf{v}_0$ is some constant vector depending on initial conditions) provided that we can make sense of the exponential of a matrix.

(Note the analogy with the first-order differential equation $\frac{dy}{dx} = ay$, which has general solution $y(x) = C \cdot e^{ax}$, where C is some real constant.)

We use the Taylor series expansion for the function e^x about $x = 0$, which is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

In the case of the matrix A, we *define*:

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots ,$$

provided this series converges, which in fact it always does. (For a more precise definition, see Curtis, Definition 34.5.)

A practical computation of e^A can be difficult, however. Fortunately, when A has at least one real eigenvalue, we can put parts (a) and (b) to work for us and write $S^{-1}AS = D + N$. Then we use some facts (only one of which you need to verify):

- $e^{A+B} = e^A \cdot e^B$, **provided that** $AB = BA$
- $S^{-1}e^A S = e^{(S^{-1}AS)}$, **for any invertible** S
- **If** $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, **then** $e^D = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}$.
- **If N is nilpotent, then there is some $m \in \mathbb{N}$ such that $N^m = 0$, and hence the expression $e^N = 1 + N + \frac{N^2}{2!} + \cdots + \frac{N^{(m-1)}}{(m-1)!}$ is a finite sum!**

Putting all of this information together enables us to find the following particular solution to the original system:

$$\mathbf{v}(t) = Se^{tD}e^{tN}S^{-1}\mathbf{v}_0,$$

where $\mathbf{v}_0 = \mathbf{v}(0)$ is a set of initial conditions.

(c) Verify the second fact above, i. e. $S^{-1}e^A S = e^{(S^{-1}AS)}$, for any invertible S.

For the moment, we will ignore the convergence questions associated with distributing multiplication across an infinite series.

We begin by proving a small lemma using mathematical induction, namely:

$$(S^{-1}AS)^n = S^{-1}A^n S.$$

This is clearly true in the $n = 1$ case, and so we assume it to be true for $n = k$ case. Then

$$\begin{aligned}
(S^{-1}AS)^{k+1} &= (S^{-1}AS)^k(S^{-1}AS), && \text{by routine factorization among matrices}\\
&= S^{-1}A^k S(S^{-1}AS), && \text{by the induction hypothesis}\\
&= S^{-1}A^k(SS^{-1})AS, && \text{by associativity of matrix multiplication}\\
&= S^{-1}A^{k+1}S, && \text{after simplification}
\end{aligned}$$

which is what we wanted to show.

Applying this to the exponential question, we get:

$$\begin{aligned}
S^{-1}e^A S &= S^{-1}(I + A + \tfrac{A}{2!} + \tfrac{A^3}{3!} + \cdots)S\\
&= S^{-1}S + S^{-1}AS + \frac{S^{-1}A^2S}{2!} + \frac{S^{-1}A^3S}{3!} + \cdots\\
&= I + (S^{-1}AS) + \frac{(S^{-1}AS)^2}{2!} + \frac{(S^{-1}AS)^3}{3!} + \cdots\\
&= e^{(S^{-1}AS)}
\end{aligned}$$

(d) Use these techniques to find the particular solution to the differential equation $y'' - y' - 6y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 1$.

With the development above, we have the equation $\dfrac{d\mathbf{v}}{dt} = A\mathbf{v}$, where $A = \left[\begin{smallmatrix} 0 & 1 \\ 6 & 1 \end{smallmatrix}\right]$. This matrix has characteristic polynomial $f_A(\lambda) = \lambda^2 - \lambda - 6$ which factors as $(\lambda - 3)(\lambda + 2)$ and has roots 3 and -2, which are then the eigenvalues of A. In other words, the diagonalized from of A will be $S^{-1}AS = \left[\begin{smallmatrix} 3 & 0 \\ 0 & -2 \end{smallmatrix}\right]$, and we must simply find S. It is a straightforward computation to discover that $\mathbf{v}_1 = \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$ is a basis vector for the eigenspace E_3 and that $\mathbf{v}_2 = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ is a basis vector for the eigenspace E_{-2}.

Thus, we may take our change of basis matrix to be $S = \left[\begin{smallmatrix} 1 & 1 \\ 3 & -2 \end{smallmatrix}\right]$, which by a short computation yields $S^{-1} = \frac{1}{5}\left[\begin{smallmatrix} 2 & 1 \\ 3 & -1 \end{smallmatrix}\right]$.

Note that since $S^{-1}AS = D + N$ is already diagonalized, we see N is the zero matrix and $e^{tN} = I$. Given the diagonal matrix D, we get the exponential $e^{tD} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$.

Putting all the pieces together, and noting that our initial conditions give the vector $\mathbf{v}_0 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, we get a final solution of:

$$\mathbf{v}(t) = Se^{tD}e^{tN}S^{-1}\mathbf{v}_0 = \left[\begin{smallmatrix}1 & 1\\3 & -2\end{smallmatrix}\right]\begin{bmatrix}e^{3t} & 0\\0 & e^{-2t}\end{bmatrix} I \cdot \frac{1}{5}\left[\begin{smallmatrix}2 & 1\\3 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = \begin{bmatrix}\frac{3}{5}e^{3t}+\frac{2}{5}e^{-2t}\\ \frac{9}{5}e^{3t}-\frac{4}{5}e^{-2t}\end{bmatrix},$$

and hence $y(t) = \frac{3}{5}e^{3t} + \frac{2}{5}e^{-2t}$.

(e) **Use these techniques to find the particular solution to the differential equation $y''-4y'+4y=0$ with initial conditions $y(0)=1$ and $y'(0)=1$.**

Once again using the development above, we have the equation $\frac{d\mathbf{v}}{dt} = A\mathbf{v}$, where $A = \left[\begin{smallmatrix}0 & 1\\-4 & 4\end{smallmatrix}\right]$. This matrix has characteristic polynomial $f_A(\lambda) = \lambda^2 - 4\lambda + 4$ which factors as $(\lambda-2)^2$ and has a double-root of 2, which is the single eigenvalue of A. As we saw in part (a), A may be upper-triangularized into the form $S^{-1}AS = \left[\begin{smallmatrix}2 & b\\0 & 2\end{smallmatrix}\right]$, and we must find an appropriate S. First one find one eigenvector to serve as the first column of S, and we discover $\mathbf{v}_1 = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ is a basis vector for the eigenspace E_2. For the second column of S, we choose (cleverly) the vector $\mathbf{v}_2 = \left[\begin{smallmatrix}3\\5\end{smallmatrix}\right]$ so that S will have determinant 1, and overall we get our change of basis matrix to be $S = \left[\begin{smallmatrix}1 & 3\\2 & 5\end{smallmatrix}\right]$, which by a short computation yields $S^{-1} = \left[\begin{smallmatrix}-5 & 3\\2 & 1\end{smallmatrix}\right]$.

Working this out to upper-triangularize A, we get

$$S^{-1}AS = \left[\begin{smallmatrix}2 & -1\\0 & 2\end{smallmatrix}\right] = D + N = \left[\begin{smallmatrix}2 & 0\\0 & 2\end{smallmatrix}\right] + \left[\begin{smallmatrix}0 & -1\\0 & 0\end{smallmatrix}\right].$$

The diagonal matrix D has exponential $e^{tD} = \begin{bmatrix}e^{2t} & 0\\0 & e^{2t}\end{bmatrix}$, and the nilpotent matrix N has exponential $e^{tN} = I + tN\left[\begin{smallmatrix}1 & -t\\0 & 1\end{smallmatrix}\right]$.

Putting all the pieces together, and noting that our initial conditions once again give the vector $\mathbf{v}_0 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, we get a final solution of:

$$\mathbf{v}(t) = Se^{tD}e^{tN}S^{-1}\mathbf{v}_0 = \left[\begin{smallmatrix}1 & 3\\2 & 5\end{smallmatrix}\right]\begin{bmatrix}e^{2t} & 0\\0 & e^{2t}\end{bmatrix}\left[\begin{smallmatrix}1 & -t\\0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}-5 & 3\\2 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = \begin{bmatrix}(1-t)e^{2t}\\(1-2t)e^{2t}\end{bmatrix},$$

and hence $y(t) = (1-t)e^{2t}$.