

MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Solutions to Final Exam (in-class portion)
January 22, 2003

1. **True or False** (28 points, 2 each)

T or **F** If V is a vector space and $S \subset V$ is a set of vectors that spans V , then S contains a basis for V .

True.

T or **F** If U and V are vector spaces over the same field F , then $(U \oplus V)/V \cong U$.

True. This is a direct consequence of the Second Isomorphism Theorem.

T or **F** If $A, B \in M_n(F)$, then $\det(AB) = \det(A) \cdot \det(B)$.

True. This is a theorem about determinants.

T or **F** $\text{sgn}((134)(25)) = +1$

False. The permutation may be written as a product of transpositions as follows:

$$(134)(25) = (14)(13)(25),$$

and hence $\text{sgn}((134)(25)) = -1$.

T or **F** The number of *odd* permutations of n elements is $n!/2$.

True. Note that a correction was made during the exam which restricted the question to the case when $n \geq 2$. In this situation, $|S_n| = n!$, and the odd and even permutations are split evenly.

- T** or **F** Every alternating multilinear form $f : V^n \rightarrow F$ is skew-symmetric.
True. This is a theorem.
- T** or **F** If $\dim(V) = m$ and $f : V^n \rightarrow F$ is an alternating form with $n > m$, then $f = 0$.
True. This is a theorem.
- T** or **F** If $A, B \in M_n(F)$ and $\text{Spec}(A) = \text{Spec}(B)$, then there is some invertible $S \in M_n(F)$ such that $A = SBS^{-1}$.
False. The matrices $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ both have eigenvalues of 1 and 2, but they are already diagonalized, and they are not similar.
- T** or **F** If V is a normed vector space with norm $\|\cdot\|$, then the function $d(x, y) = \|x - y\|$ defines a metric on V .
True.
- T** or **F** If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous and $S \subset \mathbb{R}^n$ is open, then $f(S)$ is open in \mathbb{R}^m .
False. If $U \subset \mathbb{R}^m$ is open, then $f^{-1}(U) \subset \mathbb{R}^n$ is also open. The converse is not true. For example, let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = 2$, and let $S = (0, 1)$.
- T** or **F** If $A \subset \mathbb{R}^n$ is open, then $A^\circ = A$, where A° is the interior of A .
True.
- T** or **F** If $\{A_n\}_{n=1}^\infty$ is any collection of open sets in \mathbb{R}^n , then $\bigcap_{n=1}^\infty A_n$ is open.
False. For example, consider $A_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$. Then $\bigcap_{n=1}^\infty A_n = \{0\}$, which is closed.
- T** or **F** \mathbb{Z} is closed as a subset of \mathbb{R} .
True.

T or **F** Any three non-zero vectors in \mathbb{R}^3 may be turned into an orthonormal basis via the Gram-Schmidt Orthogonalization Process.
False. The vectors must also be linearly independent for the conclusion to hold.

2. The Shift Operator (9 points, 2/3/4)

Let $\ell = \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{R}, \forall i\}$ be the vector space of all infinite sequences of real numbers. Consider the shift operator $S : \ell \rightarrow \ell$ that acts as follows: $S(a_0, a_1, a_2, a_3, \dots) = (a_1, a_2, a_3, \dots)$.

(a) Define the new operator $T : \ell \rightarrow \ell$ by the rule $T = S^2 - S$. Write down the transformation T explicitly in terms of a vector $\mathbf{v} = (a_0, a_1, a_2, \dots)$.

$$\begin{aligned} T(a_0, a_1, a_2, a_3, \dots) &= (S^2 - S)(a_0, a_1, a_2, a_3, \dots) \\ &= S^2(a_0, a_1, a_2, a_3, \dots) - S(a_0, a_1, a_2, a_3, \dots) \\ &= (a_2, a_3, a_4, a_5, \dots) - (a_1, a_2, a_3, a_4, \dots) \\ &= (a_2 - a_1, a_3 - a_2, a_4 - a_3, a_5 - a_4, \dots) \end{aligned}$$

(b) Show that every real number $\lambda \in \mathbb{R}$ is an eigenvalue for T by explicitly naming a non-zero eigenvector corresponding to it.

Let $\lambda \in \mathbb{R}$. If $T(\mathbf{a}) = \lambda \mathbf{a}$ for some $\mathbf{a} = (a_0, a_1, a_2, a_3, \dots)$, then equating both sides yields the set of equations:

$$\begin{aligned} \lambda a_0 &= a_2 - a_1 \\ \lambda a_1 &= a_3 - a_2 \\ \lambda a_2 &= a_4 - a_3 \\ \lambda a_3 &= a_5 - a_4 \end{aligned}$$

and, in general, for each $n \geq 2$,

$$\lambda a_{n-2} = a_n - a_{n-1}$$

In other words, letting $a_0 = 1$ and $a_1 = 0$ (for example), we generate an eigenvector for λ with the terms $a_n = a_{n-1} + \lambda a_{n-2}$. The first few terms give: $\mathbf{a} = (1, 0, \lambda, \lambda, \lambda^2 + \lambda, 2\lambda^2 + \lambda, \dots)$.

- (c) Let ℓ_1 be the eigenspace for T corresponding to the eigenvalue 1. Find a basis for ℓ_1 , and prove that it is a basis.

The process from part (b) above shows that a vector $\mathbf{c} = (c_0, c_1, c_2, c_3, \dots)$ will be in ℓ_1 provided that $c_n = c_{n-1} + c_{n-2}$, for each $n \geq 2$. This leaves two degrees of freedom to choose c_0 and c_1 .

We let $\mathbf{a} = (1, 0, 1, 1, 2, 3, 5, 8, \dots)$ where $a_n = a_{n-1} + a_{n-2}$, for each $n \geq 2$, and $\mathbf{b} = (0, 1, 1, 2, 3, 5, 8, \dots)$ where $b_n = b_{n-1} + b_{n-2}$, for each $n \geq 2$,

These two vectors are linearly independent because neither is a scalar multiple of the other. To show that they span ℓ_1 , consider any $\mathbf{c} = (c_0, c_1, c_2, c_3, \dots)$ satisfying the condition $c_n = c_{n-1} + c_{n-2}$, for every $n \geq 2$. Then $\mathbf{c} = c_0\mathbf{a} + c_1\mathbf{b}$, and so $\mathbf{c} \in \text{span}\{\mathbf{a}, \mathbf{b}\}$.

3. Cramer's Rule (16 points, 2/4/3/3/4)

Consider the vector space $V = F^n$ and the invertible linear transformation $A : V \rightarrow V$. If $\mathbf{b} \in V$ is some fixed vector, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution \mathbf{x} , given as follows:

$$\text{If } A = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix}, \text{ let } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

$$\text{where } x_i = (\det A)^{-1} \cdot \det \begin{bmatrix} | & & | & | & | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_{i-1} & \mathbf{b} & \mathbf{v}_{i+1} & \cdots & \mathbf{v}_n \\ | & & | & | & | & & | \end{bmatrix}.$$

Prove and apply this result in the following steps:

- (a) Write \mathbf{b} as a linear combination of the columns of A . (Why can this be done?)

Since A is invertible, its columns are linearly independent, and since any set of n linearly independent vectors in n -dimensional space forms a basis, we see that we may write any vector $\mathbf{b} \in V$ uniquely as a linear combination of the columns of A . In particular, we let:

$$\mathbf{b} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

- (b) If $D : V^n \rightarrow F$ is the non-zero alternating form used to define the determinant, evaluate the expression $D(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{b}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n)$, in terms of your linear combination from part (a).

Naming the expression y_i for the moment, we have

$$\begin{aligned} y_i &= D(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{b}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n) \\ &= D(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, b_1\mathbf{v}_1 + \dots + b_n\mathbf{v}_n, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n) \\ &= b_1D(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_1, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n) + \dots + b_nD(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_n, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n) \\ &= b_iD(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n) \\ &= b_i \cdot \det(A), \end{aligned}$$

where the third equality follows from the multilinearity condition on D and the fourth follows from the alternating condition. Finally, the determinant of A is defined in terms of D in exactly the manner specified by the last equality.

- (c) Show that the vector \mathbf{x} as defined above satisfies the equation $A\mathbf{x} = \mathbf{b}$.

Using the notation from part (b), the transpose (to avoid column vectors for a moment) of our vector \mathbf{x} can be represented as $(x_1, \dots, x_n) = (\det A)^{-1}(y_1, \dots, y_n) = (b_1, \dots, b_n)$. Then

$$A\mathbf{x} = \begin{bmatrix} \vdots & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = b_1\mathbf{v}_1 + \dots + b_n\mathbf{v}_n = \mathbf{b}.$$

- (d) Show that this \mathbf{x} is the *unique* solution to the equation $A\mathbf{x} = \mathbf{b}$.

Using our typical uniqueness argument, suppose \mathbf{y} is some other vector satisfying the equation $A\mathbf{y} = \mathbf{b}$. Then by linearity, we may subtract and write $A\mathbf{x} - A\mathbf{y} = \mathbf{0}$, or in other words, $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$, which implies that $\mathbf{x} - \mathbf{y} \in \text{Ker}(A)$. Since A is invertible, its kernel is trivial, and hence $\mathbf{x} - \mathbf{y} = \mathbf{0}$, or in other words, $\mathbf{x} = \mathbf{y}$.

- (e) Use Cramer's Rule to solve the system of equations:

$$\begin{aligned} x + 2y + 3z &= 1 \\ & y + 4z = 0 \\ x & - 6z = -1 \end{aligned}$$

Straightforward computation of the relevant determinants reveals that there is a unique solution of $x = 11$, $y = -8$, and $z = 2$.

4. Open and Closed Sets in Euclidean Space (9 points, 3 each)

Let V be a Euclidean space.

(a) If $A \subset V$, define what it means for A to be an *open set*.

A is *open* if, given any $\mathbf{x} \in A$, there is some $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subset A$. Alternatively, A is open if every point of A is an interior point.

(b) If $B \subset V$ and $\mathbf{x} \in V$, define what it means for \mathbf{x} to be a *limit point* of B .

The point \mathbf{x} is a *limit point* of B if there exists a sequence of points $\{\mathbf{b}_i\}_{i=1}^\infty$ that converges to \mathbf{x} with $\mathbf{b}_i \in B$, for every i . Alternatively, we can say that \mathbf{x} is a limit point of B if, for any $\varepsilon > 0$, $B \cap B_\varepsilon(\mathbf{x}) \neq \emptyset$.

(c) If $V = \mathbb{R}^2$ and $C = \{(\frac{1}{n}, \frac{1}{m}) | n, m \in \mathbb{N}\}$, then find ∂C , the *boundary* of C .

$$\partial C = C \cup \{(0, \frac{1}{m}) | m \in \mathbb{N}\} \cup \{(\frac{1}{n}, 0) | n \in \mathbb{N}\} \cup \{(0, 0)\}$$

5. Continuity and the Topology Euclidean Space (14 points, 3/3/8)

Let $M_n(\mathbb{R})$ be the n^2 -dimensional Euclidean space of $n \times n$ matrices with real entries, and take the inner product to be the usual one, where $M_n(\mathbb{R})$ is naturally isomorphic to \mathbb{R}^{n^2} .

(a) Define what it means for $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to be continuous at $\mathbf{x} \in \mathbb{R}^n$.

The function f is continuous at $\mathbf{x} \in \mathbb{R}^n$ provided that, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $\|\mathbf{x} - \mathbf{a}\| < \delta$ implies that $\|f(\mathbf{x}) - f(\mathbf{a})\| < \varepsilon$.

(b) Let $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) = 1\}$. Show that $SL_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$.

We showed on the take-home final (question # 1(a)) that the determinant function was continuous (since it was the sum of scalar multiples of products of coordinate projection functions, which are themselves continuous).

Next, we note that the set $A = \{1\}$ is a closed subset of \mathbb{R} . (Its complement is open because, given any $y \in \mathbb{R} \setminus A$, choose $\varepsilon = |y - 1|$, and we have $B_\varepsilon(y) \cap A = \emptyset$.)

Finally, we have a theorem the inverse image of a closed set under a continuous map is also closed, and so $SL_n(\mathbb{R}) = (\det)^{-1}(A)$ is closed.

- (c) **In this part, for simplicity, we specialize to the case $n = 2$. Let X be the collection of diagonalizable matrices in $M_2(\mathbb{R})$. Is X open, closed, or neither? Explain.**

We show that X is *neither* open nor closed as follows:

1. The set X is not open because we can produce a sequence of diagonalizable matrices which converge to a matrix which is not diagonalizable. In particular, let $A_n = \begin{bmatrix} 1 & 1 \\ 0 & 1 + \frac{1}{n} \end{bmatrix}$. Each A_n is diagonalizable because it has two distinct eigenvalues, namely 1 and $1 + \frac{1}{n}$, and clearly, $A_n \rightarrow A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Note that A is not diagonalizable because its unique eigenvalue is 1 while the corresponding eigenspace is $E_1 = \text{span}\{\mathbf{e}_1\}$, which is one-dimensional.

2. The set X^c is not open (and hence X is not closed) because we can produce a sequence of non-diagonalizable matrices which converge to a matrix which is diagonalizable. In particular, let $B_n = \begin{bmatrix} 1 & \frac{1}{n} \\ 0 & 1 \end{bmatrix}$. Each B_n is non-diagonalizable for the same reason as the matrix A above, and clearly, $B_n \rightarrow B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is not only diagonalizable but diagonal.

6. Linear Transformations on Euclidean Space (12 points, 4 each)

Let $V = \mathbb{R}^3$, and let $W = \text{span}\{(1, 2, 2), (0, 3, 6)\}$.

Recall throughout that if $\mathbf{u}, \mathbf{v} \in V$, then the projection of \mathbf{u} in the direction of \mathbf{v} is given by: $\text{Proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$, and if \mathbf{v} is a unit vector, then this simplifies to: $\text{Proj}_{\mathbf{v}}\mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{v}$.

(a) Find an orthonormal basis for W .

Let $\mathbf{w}_1 = (1, 2, 2)$ and $\mathbf{w}_2 = (0, 3, 6)$. To find an ONB for W we use the Gram-Schmidt Orthogonalization Process.

$$\text{Let } \mathbf{v}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{(1, 2, 2)}{3} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right).$$

$$\text{Let } \mathbf{x}_2 = \mathbf{w}_2 - \text{Proj}_{\mathbf{v}_1}\mathbf{w}_2 = (0, 3, 6) - 6\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = (-2, -1, 2).$$

$$\text{Let } \mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \frac{(-2, -1, 2)}{3} = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right).$$

Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an ONB for W .

(b) In terms of the standard basis for V , write the matrix for the linear transformation $P : V \rightarrow V$ that is projection onto the subspace W .

For any $\mathbf{u} \in V$, we have $P(\mathbf{u}) = \text{Proj}_{\mathbf{v}_1}\mathbf{u} + \text{Proj}_{\mathbf{v}_2}\mathbf{u}$. Alternatively, note that $\mathbf{v}_3 = \left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ makes the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ into an ONB for V , and hence $P(\mathbf{u}) = \mathbf{u} - \text{Proj}_{\mathbf{v}_3}\mathbf{u}$.

In either case, a straightforward computation yields

$$P(\mathbf{e}_1) = \left(\frac{5}{9}, \frac{4}{9}, -\frac{2}{9}\right), P(\mathbf{e}_2) = \left(\frac{4}{9}, \frac{5}{9}, \frac{2}{9}\right), \text{ and } P(\mathbf{e}_3) = \left(-\frac{2}{9}, \frac{2}{9}, \frac{8}{9}\right),$$

$$\text{and hence, in terms of the standard basis, } P = \frac{1}{9} \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix}.$$

(c) Describe the linear transformation $2P - I$ geometrically, and show that $(2P - I)^2 = I$.

Geometrically, the transformation $A = 2P - I$ is a reflection through the plane W , and it is natural that a reflection composed with itself is the identity. Algebraically, we showed on the homework that the square of any involution is the identity. (Written out, recall that $P^2 = P$ for any projection, and hence $A^2 = (2P - I)^2 = 4P^2 - 4P + I = 4P - 4P + I = I$.) Finally, one can simply do the multiplication using the matrix for P found in part (b) to see that $A^2 = I$.