MATH 23a, FALL 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Final Exam Preview Problems

Directions: What follow are six questions for you to work on in preparation for our in-class final examination, which will take place on Saturday, January 24, from 2:15–5:15 P.M. in Lecture Hall C of the Science Center. These are longer questions that would be difficult in the environment of an exam with strict time constraints, but if you have seen them before, they should be quite reasonable. You should expect that at least one and probably two of these questions will appear on the in-class final. The questions may not be taken verbatim from this set, they will be very close, and I reserve the right to include parts from any of these questions on the final, in addition to the one/two that will appear in their entirety.

You are encouraged to work on these problems with each other, with the course assistants, or with me, and you may consult any references you like. Keep in mind, however, that you will need to be able to reproduce whatever ideas you gather from other sources, so the premium is on understanding the solutions.

1. Nilpotent Matrices

Let $V = \mathbb{R}^n$, and let $A : V \to V$. We say that A is *nilpotent* if there is some $m \in \mathbb{N}$ such that $A^m = 0$, and we say that m is the *degree* of nilpotency if $A^m = 0$ but $A^{(m-1)} \neq 0$.

(a) Let $A : V \to V$ be nilpotent of degree m. Let $\mathbf{v} \in V$ be such that $A^{(m-1)}\mathbf{v} \neq \mathbf{0}$. Show that $\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \ldots, A^{(m-1)}\mathbf{v}\}$ is a linearly independent set in V. Conclude that $m \leq n$.

(b) Exhibit examples of nilpotent matrices of degrees 0,1,2, and 3.

(c) Let $A : V \to V$ be nilpotent. Find all the eigenvalues of A.

(d) Show that if A is nilpotent, then $I + A$ is invertible.
(If A is nilpotent, then $I + A$ is called *unipotent*.)

2. Long Exact Sequences.

Let $V_1, \ldots, V_n$ be finite-dimensional vector spaces over the same field, and suppose we have the following sequence of linear maps:

$$\{\mathbf{0}\} \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \quad \cdots \quad \longrightarrow V_n \longrightarrow \{\mathbf{0}\}$$

where $\varphi_i : V_i \longrightarrow V_{i+1}$, for $0 \leq i \leq n$.
(Here, we let $V_0 = V_{n+1} = \{\mathbf{0}\}$ for consistency in the subscripts.)

Such a collection of vector spaces and linear maps is known as a *(long) exact sequence* when they satisfy the relationship:

$$Im(\varphi_i) = Ker(\varphi_{i+1}), \quad \text{for } 0 \leq i \leq n.$$

Show that $\displaystyle\sum_{i=0}^{n} (-1)^i \cdot dim(V_i) = 0.$

3. Orthogonal Matrices.

Let $V = \mathbb{R}^n$ be Eulcidean space, where the usual inner product may be expressed as $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^t \mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are thought of as $n \times 1$ matrices, in coordinates with respect to the standard basis. In other words, if $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$, then

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + \cdots + v_n w_n.$$

Recall that we define the collection of invertible linear transformations, called the *general linear group*, to be:

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) |\ A \text{ is invertible}\}.$$

Now, we further define a subgroup of the general linear group, called the *orthogonal group*, to be:

$$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) |\ A^t A = I\}.$$

(a) Show that $A \in O_n(\mathbb{R})$ if and only if A is inner-product preserving. (See HW #9D.)

(b) Show that the columns of any $A \in O_n(\mathbb{R})$ form an orthonormal basis for V by showing that:

   i. The columns of any $A \in O_n(\mathbb{R})$ are vectors of norm 1.

   ii. The columns of any $A \in O_n(\mathbb{R})$ are mutually orthogonal.

(c) If $A \in O_n(\mathbb{R})$, find all possible values of $det(A)$.

(d) For each possible answer in part (c), find the general form of a matrix in $O_2(\mathbb{R})$ with that determinant.

(e) For each possible answer in part (d), find all possible eigenvalues for those matrices in $O_2(\mathbb{R})$.

4. Cramer's Rule

Consider the vector space $V = F^n$ and the invertible linear transformation $A : V \to V$. If $\mathbf{b} \in V$ is some fixed vector, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$, given as follows:

If $A = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix}$, let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$,

where $x_i = (\det A)^{-1} \cdot \det \begin{bmatrix} | & & | & | & | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_{i-1} & \mathbf{b} & \mathbf{v}_{i+1} & \cdots & \mathbf{v}_n \\ | & & | & | & | & & | \end{bmatrix}$.

Prove and apply this result in the following steps:

(Hint: You might do part (e) first to get a feel for this problem.)

(a) Write $\mathbf{b}$ as a linear combination of the columns of A. (Why can this be done?)

(b) If $D : V^n \to F$ is the non-zero alternating form used to define the determinant, evaluate the expression $D(\mathbf{v}_1, \cdots, \mathbf{v}_{i-1}, \mathbf{b}, \mathbf{v}_{i+1}, \cdots, \mathbf{v}_n)$, in terms of your linear combination from part (a).

(c) Show that the vector $\mathbf{x}$ as defined above satisfies the equation $A\mathbf{x} = \mathbf{b}$.

(d) Show that this $\mathbf{x}$ is the *unique* solution to the equation $A\mathbf{x} = \mathbf{b}$.

(e) Use Cramer's Rule to solve the system of equations:

$$\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 1 \\ & & y & + & 4z & = & 0 \\ x & & & - & 6z & = & -1 \end{array}$$

5. Orthogonal Complement of a Subspace

Let V be an inner-product space, which for this problem you may assume is finite-dimensional. For a subspace U of V we define the *orthogonal complement* of U as

$$U^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{u} \rangle = 0,\ \forall \mathbf{u} \in U\}.$$

(a) Show that $V = U \oplus U^{\perp}$.

Now we know from part (a) that each $\mathbf{v} \in V$ can be written uniquely as $\mathbf{v} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$ and $\mathbf{w} \in U^{\perp}$. We define a map $P_U : V \to V$ by the rule $P_U(\mathbf{v}) = \mathbf{u}$, where $\mathbf{v} = \mathbf{u} + \mathbf{w}$ as above.

(b) Show that P_U is linear.
(c) Show that $Im(P_U) = U$.
(d) Show that $Ker(P_U) = U^{\perp}$.
(e) Show that $||P_U(\mathbf{v})|| \leq ||\mathbf{v}||$ for all $\mathbf{v} \in V$.

6. Jordan Blocks

We have seen in class that some, but not all, matrices are diagonalizable. For a matrix that is not, one of the next best results along these lines would be to be able to put such a matrix in "Jordan Canonical Form." In this problem, we tackle one of the building blocks for this result.

Let $V = \mathbb{R}^3$, and let $A : V \to V$. Suppose that the characteristic polynomial of A is $p_A(\lambda) = (\alpha - \lambda)^3$. (See HW #8B.) Show that exactly one of the following possibilities must hold:

- A is diagonalizable. (What is the diagonalized form of A?)
- There is a basis for V with respect to which

$$A = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}.$$

- There is a basis for V with respect to which

$$A = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{bmatrix}.$$