MATH 23a, FALL 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 2
Due: October 3, 2003

1. Read Sections 3.1–3.3 from Schneider and Barker and 1.1–1.2 from Edwards.

2. (A) Using the uniqueness of additive inverses in a ring (see HW #1.8), prove that $(-a)(-b) = ab$. *(Hint: Show that each of the two expressions is the additive inverse of some third expression.)*

3. (A) If V is a vector space and $\mathbf{v} \in V$, show that $(-1)\mathbf{v} = -\mathbf{v}$.

   (Note that there really is something to do here. The expression on the right is the additive inverse of the vector, and the one on the left is the vector multiplied by the scalar -1, and these are not *a priori* the same thing.)

4. (*) Show that the set of ordered triples (x, y, z) of real numbers satisfying the equations $2x + y = 0$ and $3y - z = 0$ form a vector space over the field $\mathbb{R}$.

5. (B and C) Set (B) will be parts (a),(b), and (c).

   Set (C) will be parts (d) and (e).

   Let p be a fixed prime number, and let $V = (\mathbb{Z}/p\mathbb{Z})^n$.

   (a) How many vectors are in the vector space V?

   (b) In the case $n = 2$ and arbitrary $p > 2$, show that any vector may be written as a linear combination of the two vectors $(1, 2)$ and $(1, 1)$.

   (c) In the case $n = 2$ and $p = 7$, show that the vectors $(1, 6)$, $(2, 4)$, and $(3, 3)$ are not linearly independent.

   (d) In the case $n = 2$ and $p = 7$, find an explicit example (writing down all vectors) of a non-trivial subspace (that is, not $\{\mathbf{0}\}$, and not V).

   (e) How many subspaces does $V = (\mathbb{Z}/7\mathbb{Z})^2$ have?
   (Hint: Consider a generalization of part (d).)

   (f) (deferred)
   How many distinct one-dimensional subspaces does V have?

   (g) (deferred)
   How many distinct two-dimensional subspaces does V have?
   (Assuming $n \geq 2$, of course.)

6. (D) Recall the vector space $C[0,1] = \{f : [0,1] \to \mathbb{R} | f \text{ is continuous}\}$.

   Now, for each $c \in \mathbb{R}$, consider the set of vectors defined by

$$V(c) = \{f : [0,1] \to \mathbb{R} | f \text{ is continuous, and } f(0) = c\}.$$

   For which values of c is $V(c)$ a subspace of $C[0,1]$? Explain.

7. (*) Find an infinite set of linearly independent vectors in $C[0,1]$, not including any polynomials. (Think also about what you would need to do in order to *prove* that your set was linearly independent.)

8. (deferred) Show that if V is a vector space with $dim(V) = n$, then any collection of $n+1$ vectors in V is linearly dependent.

9. (deferred) Let S be some set with finite cardinality n. (For simplicity, you may assume that $S = \{1, 2, \ldots, n\}$.) Find a basis for the vector space of functions $V = \{f : S \to \mathbb{R}\}$, where addition and scalar multiplication are defined as for functions of one real variable.

10. (E) Let V be a vector space over the field F defined as all infinite sequences of elements of F:

$$V = \{(a_0, a_1, a_2, \ldots) \mid a_i \in F, \forall i \geq 0\}$$

   For $k = 1, 2$, we define the two subspaces:

$$\ell^k = \left\{ (a_0, a_1, a_2, \ldots) \;\middle|\; a_i \in F, \forall i \geq 0, \text{and} \sum_{n=0}^{\infty} |a_n|^k \quad \text{converges} \right\}$$

   (a) (*Not required*) Show/convince yourself that V is a vector space.

   (b) Let $F = \mathbb{R}$. Show that ℓ^2 is a subspace of V by showing that the addition of vectors and scalar multiplication are closed operations. That is, if $\mathbf{u}, \mathbf{v} \in \ell^2$ and $a \in \mathbb{R}$, then $\mathbf{u} + \mathbf{v} \in \ell^2$ and $a\mathbf{u} \in \ell^2$. *(Hint: You may cite appropriate results from Calculus.)*

   (c) Let $F = \mathbb{R}$. Show that $\ell^1 \subset \ell^2$. Show by example that $\ell^1 \neq \ell^2$.

   (d) Let $F = \mathbb{Z}/2\mathbb{Z}$. Note that by "converges", we mean (as always!) that the sequence of partial sums has a limit.

   Give an explicit condition for deciding whether of not a particular sequence is in ℓ^1 or ℓ^2. (What must happen for such an infinite series to converge? Recall that the limits and sums are being taken in the finite field!)