

MATH 23b, SPRING 2003  
THEORETICAL LINEAR ALGEBRA  
AND MULTIVARIABLE CALCULUS  
Homework Assignment # 2  
Due: February 14, 2003

Homework Assignment #2 (Final Version)

1. Read Fitzpatrick, Section 12.5 and Chapter 13 (especially sections 1-3).
2. (a) (A) Let  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R} \rightarrow \mathbb{R}^n$  be two differentiable curves, with  $f'(t) \neq 0$  and  $g'(t) \neq 0$  for all  $t \in \mathbb{R}$ . Suppose that  $\mathbf{p} = f(s_0)$  and  $\mathbf{q} = g(t_0)$  are closer than any other pair of points on the two curves. Prove that the vector  $\mathbf{p} - \mathbf{q}$  is orthogonal to both velocity vectors  $f'(s_0)$  and  $g'(t_0)$ .  
*(Hint: The point  $(s_0, t_0)$  must be a critical point for the function  $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $\rho(s, t) = |f(s) - g(t)|^2$ .)*  
(b) Apply the result of part (a) to find the closest pair of points of the “skew” straight lines in  $\mathbb{R}^3$  defined by  $f(s) = (s, 2s, -s)$  and  $g(t) = (t + 1, t - 2, 2t + 3)$ .
3. (B) Consider a particle which moves on a circular helix in  $\mathbb{R}^3$  with position vector given by (all scalars non-zero):

$$\gamma(t) = (a \cos \omega t, a \sin \omega t, b\omega t).$$

- (a) Show that the speed of the particle is constant.
- (b) Show that the velocity vector makes a constant nonzero angle with the  $z$ -axis.
- (c) If  $t_1 = 0$  and  $t_2 = \frac{2\pi}{\omega}$ , notice that  $\gamma(t_1) = (a, 0, 0)$  and  $\gamma(t_2) = (a, 0, 2\pi b)$ , so the vector  $\gamma(t_2) - \gamma(t_1)$  is vertical. Conclude that the equation

$$\gamma(t_2) - \gamma(t_1) = (t_2 - t_1)\gamma'(\tau)$$

cannot hold for any  $\tau \in (t_1, t_2)$ . Thus the Mean Value Theorem does not hold for vector-valued functions.

4. (C) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by  $f(x, y) = \frac{x^2 y}{x^2 + y^2}$  unless  $x = y = 0$  and  $f(0, 0) = 0$ .
- (Not required) Graph  $f$  using Mathematica! (*The picture of this function was distributed in class, but it will be useful for you to be able to generate it yourself.*)
  - Show that  $D_{\mathbf{v}}f(0, 0)$  exists for all  $\mathbf{v} \in \mathbb{R}^2$  by direct computation. (*Hint: You should conclude that  $D_{\mathbf{v}}f(0, 0) = f(\mathbf{v})$ .*)
  - Show that  $f$  satisfies the homogeneous relation  $f(t\mathbf{v}) = tf(\mathbf{v})$  for all  $t \in \mathbb{R}$  and all  $\mathbf{v} \in \mathbb{R}^2$ .
  - Show that any *differentiable* function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying the homogeneous relation  $g(t\mathbf{v}) = tg(\mathbf{v})$ ,  $\forall t \in \mathbb{R}, \forall \mathbf{v} \in \mathbb{R}^n$  and  $g(\mathbf{0}) = 0$  also satisfies the relation  $g(\mathbf{v}) = \nabla g(\mathbf{0}) \cdot \mathbf{v}$  for all  $\mathbf{v} \in \mathbb{R}^n$  and hence must be *linear*.
  - Conclude that  $f$  possesses directional derivatives in all directions at  $(0, 0)$ , but that  $f$  is *not* differentiable at  $(0, 0)$ .
5. (D) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by  $f(x, y) = x^2 \sin(1/x) + y^2$  for  $x \neq 0$  and  $f(0, y) = y^2$ .
- Show that  $f$  is continuous at  $(0, 0)$ .
  - Find the partial derivatives of  $f$  at  $(0, 0)$ .
  - Show that  $f$  is differentiable at  $(0, 0)$ .
  - Show that  $D_1 f$  is *not* continuous at  $(0, 0)$ .
6. (D) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by  $f(x, y) = (\sin(x - y), \cos(x + y))$ . Find the equations of the tangent plane in  $\mathbb{R}^4$  to the graph of  $f$  at the point  $(\frac{\pi}{4}, \frac{\pi}{4}, 0, 0)$ .
7. (Deferred to next week)

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Prove that if  $f$  is differentiable at  $\mathbf{a}$ , then  $f$  is continuous at  $\mathbf{a}$ .