Math 23b, Spring 2003

Problem Set 2, Part B
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Problem 3: *Consider a particle which moves on a circular helix in* $\mathbb{R}^3$ *with position vector given by (all scalars are non-zero):*

$$\gamma(t) = (a\cos\omega t, a\sin\omega t, b\omega t).$$

(a) Show that the speed of the particle is a constant.
(b) Show that the velocity vector makes a constant non-zero angle with the z*-axis.*
(c) If $t_1 = 0$ *and* $t_2 = \frac{2\pi}{\omega}$*, notice that* $\gamma(t_1) = (a, 0, 0)$ *and* $\gamma(t_2) = (a, 0, 2\pi b)$*, so the vector* $\gamma(t_2) - \gamma(t_1)$ *is vertical. Conclude that the equation*

$$\gamma(t_2) - \gamma(t_1) = (t_2 - t_1)\gamma'(\tau)$$

cannot hold for any $\tau \in (t_1, t_2)$*. Thus the Mean Value Theorem does not hold for vector-valued functions.*

Proof. (a) Clearly $\gamma'(t) = (-a\omega\sin(\omega t), a\omega\cos(\omega t), b\omega)$. Then the speed of the particle is $||\gamma'(t)|| = \sqrt{a^2\omega^2\sin^2(\omega t) + a^2\omega^2\cos^2(\omega t) + b^2\omega^2} = |\omega|\sqrt{a^2+b^2}$, which clearly is a constant. Remark: Many people made a mistake here by missing the absolute value. At the end ω might be negative and in general $\sqrt{\omega^2} = |\omega|$.

(b) Let's consider the unit vector on the z-axis: $z = (0, 0, 1)$ and let θ be the angle between $\gamma'(t)$ and z. We need to show that θ is a non-zero constant. To do that let's consider the scalar product of $\gamma'(t)$ and z. On one hand it is $||\gamma'(t)|| \cdot ||z|| \cos\theta = |\omega|\sqrt{a^2+b^2}\cos\theta$ and on the other it equals $b\omega$. Therefore $\cos\theta = \frac{b\omega}{|\omega|\sqrt{a^2+b^2}}$, which clearly is a constant. Moreover, it is non-zero since if it were zero we directly get that $a = 0$, which is not the case.

(c) The equation $\gamma(t_2) - \gamma(t_1) = (t_2 - t_1)\gamma'(\tau) \Leftrightarrow (0, 0, 2\pi b) = \frac{2\pi}{\omega}\gamma'(\tau)$ cannot hold for any $\tau \in (t_1, t_2)$ since the left-hand side is vertical, i.e. makes a zero angle with the z-axis, whereas the right-hand side, as we showed in (b), makes a non-zero angle with the z-axis. Remark: That the given equation cannot hold for any $\tau \in (t_1, t_2)$ can also be shown via a direct calculation, as many people did, but it requires some work, which is avoided in this solution. $\square$