**Math 23a, 2002.**
Solution Set 2, Question 4.
Joshua Reyes

Question 4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{x^2 y}{x^2 + y^2}$ unless $x = y = 0$ and $f(0, 0) = 0$.

(a) (Not required) Graph f using Mathematica!

(b) Show that $D_v f(0, 0)$ exists for all $v \in \mathbb{R}^2$ by direct computation. (*Hint: You should conclude that $D_v f(0,0) = f(v)$.*)

(c) Show that f satisfies the homogeneous relation $f(tv) = tf(v)$ for all $t \in \mathbb{R}$ and all $v \in \mathbb{R}^2$.

(d) Show that any differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying the homogeneous relation $g(tv) = tg(v), \forall t \in \mathbb{R}, \forall v \in \mathbb{R}^n$ and $g(0) = 0$ also satisfies the relation $g(v) = \nabla g(0) \cdot v$ for all $v \in \mathbb{R}^n$ and hence must be *linear*.

(e) Conclude that f possesses directional derivatives in all directions at $(0,0)$, but that f is *not* differentiable at $(0,0)$.

Answer.
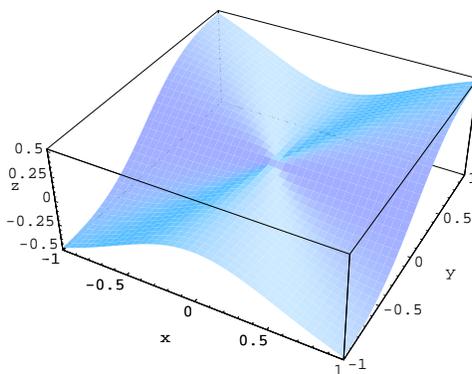
(a) To make this graph in Mathematica, I typed

```
Plot3D[x^2*y/(x^2+y^2), {x,-1,1}, {y,-1,1},
       AmbientLight->Hue[.65], PlotPoints->50, Mesh->False,
       AxesLabel->{"x", "y", "z"}]
```

(b) Let's try out the good ol' limit and compute. First set $v = (x, y)$ where neither x or $y = 0$.

$$D_v f(0,0) = \lim_{h \to 0} \frac{f(hx, hy) - f(0,0)}{h} = \lim_{h \to 0} \frac{(h^2 x^2) hy}{h(h^2 x^2 + h^2 y^2)} = \lim_{h \to 0} \frac{x^2 y}{x^2 + y^2} = f(x, y).$$

And hey, if either $x = 0$ or $y = 0$ then $D_v f(0,0) = 0$ for sure since the numerator goes away.

(c) Suppose we're still playing with the same v from up above, then

$$f(tv) = \frac{t^2 x^2 t y}{t^2 x^2 + t^2 y^2} = \frac{t^3}{t^2} \frac{x^2 y}{x^2 + y^2} = tf(v).$$

Again, if either $x = 0$, $y = 0$ or $t = 0$ then it looks pretty true, too.

(d) So we've already shown this basically. Just reuse parts (b) and (c) at the same time and call on the definition of the directional derivative. See, 'cause

$$D_v g(0) = \lim_{h \to 0} \frac{g(hv) - g(0)}{h} = \lim_{h \to 0} \frac{hg(v)}{h} = g(v).$$

1

Because g is homogeneous and $g(0) = 0$, that h swings around and out of the limit. Meanwhile, back at the ranch, we know that $D_v g(0) = \nabla g(0) \cdot v$. Put it all together and what've you got?

$$g(v) = \nabla g(0) \cdot v.$$

(e) Well, in part (a) we showed that the directional derivatives exist. And it's not too bad to see that the partials $D_1 f$ and $D_2 f$ are zero. Unfortunately, if we try going in the direction of $(1, 1)$ we get something else. That means f isn't linear despite its being homogeneous. But then part (d) kicks in to tell us that f isn't differentiable at $(0, 0)$.