Math 23a, 2002.
Solution Set 2, Question 5.
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Question 5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 \sin(1/x) + y^2$ for $x \neq 0$ and $f(0, y) = y^2$.

(a) Show that f is continuous at $(0, 0)$.
(b) Find the partial derivatives of f at $(0, 0)$.
(c) Show that f is differentiable at $(0, 0)$.
(d) Show that $D_1 f$ is *not* continuous at $(0, 0)$.

Answer. (a) For f to be continuous at $(0, 0)$, we've got to make sure the limit and the value of f agree; you know, that

$$\lim_{x,y \to 0} f(x, y) = 0.$$

Good thing that $\sin(1/x)$ is bounded above by 1 and below by -1. So f itself lives in the inequality

$$y^2 - x^2 \leq x^2 \sin(1/x) + y^2 \leq x^2 + y^2.$$

And both those extremes are 0 at $(0, 0)$, so that middle term must be, too. Got to love the squeeze theorem, eh?

(Hey, a lot of you referenced the *sandwich theorem*, which is different. At least the *ham* sandwich theorem is. Maybe we'll prove it at the very, very, very end of the semester if we've learnt enough. Well, maybe a baby version of it, anyway. Just thought you should know there's difference.)

(b) If we step back and compute all the directional derivatives, we can basically read off the partials. So, let's go: for $v = (x, y)$.

$$D_v f(0, 0) = \lim_{h \to 0} \frac{f(hx, hy) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h^2 x^2 \sin \frac{1}{hx} + h^2 y^2}{h} = \lim_{h \to 0} hx^2 \sin \frac{1}{hx} + hy^2.$$

But something bounded times something that goes to zero is zero. Since all the directionals go to zero, the partials must, too.

(c) Since all the directional derivatives are zero, f itself is differentiable with derivative zero.

(d) To find $D_1 f(x, y)$ just hold y constant and do like you did in single variable calculus.

$$D_1 f(x, y) = 2x \sin(1/x) - \cos(1/x).$$

Again, the first term is cool since the limit is bounded and goes to zero as x does. But that cosine bit is a little wilder. Take something sneaky like $x_n = 1/(2\pi n)$ and $\tilde{x}_n = 1/((2n + 1)\pi)$. As $n \to \infty$, both x_n and $\tilde{x}_n \to 0$, but $\cos(1/x_n)$ and $\cos(1/\tilde{x}_n)$ don't match up. For shame. $D_1 f$ can't be continuous.