Math 23a, 2002.
Solution Set 2, Questions 5-6.
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Question 5. Considering the real numbers as defined by equivalence classes of Cauchy sequences of rational numbers, name the equivalence class that acts as the multiplicative identity, and verify that it does.

Answer. This is another problem where you can sort of guess the answer, claim that it is and then show that you're right. These are among me favourite types of problems. Why you ask, because proving the existence of things can get really tricky. Finding them is usually much, much harder. But if you already have it, it's not so bad. So, I claim that $[\{1\}]$, that is the constant 1, is the multiplicative identity of the reals. So, for any $[\{a_n\}] \in \mathbb{R}$

$$[\{a_n\}] \cdot [\{1\}] = [\{a_n \cdot 1\}] = [\{1 \cdot a_n\}] = [\{a_n\}]$$

because multiplication in $\mathbb{Q}$ is commutative and because $1 \in \mathbb{Q}$ is the multiplicative identity. The sequence $[\{1\}]$ is Cauchy by inspection.

Question 6. Considering the real numbers as defined by equivalence classes of Cauchy sequences of rational numbers, prove the existence of multiplicative inverses (for elements other than the additive identity).

Answer. Given $[\{a_n\}] \in \mathbb{R}$ and $[\{a_n\}] \neq 0$ we'd like to say that the inverse is $[\{1/a_n\}]$, but some of the a_n could be zero. So assume that only finitely many a_n are zero. Otherwise if infinitely many are zero, since our sequence is Cauchy, we can find an N such that for all n, $m > N$, we have $|a_n - a_m| < \epsilon$. Now consider $n > N$ and $a_n = 0$. Plugging in we're left with $|0 - a_m| < \epsilon$ whenever $m > N$. But then the whole sequence converges to 0 and $[\{a_n\}]$ was 0 all along. *Contradiction!* There must only be a finite number of zeros in our sequence.

Now since only finite terms in the sequence are zero, we can pick another representative $\{b_n\}$ in $[\{a_n\}]$ by moving far enough down the sequence past all the zeros. So we claim that $[\{1/b_n\}]$ is the multiplicative inverse of $[\{b_n\}]$. And we can show this.

$$[\{b_n\}] \cdot [\{1/b_n\}] = [\{1/b_n\}] \cdot [\{b_n\}] = [\{b_n/b_n\}] = [\{1\}] = 1.$$

But there's a catch! We don't know that this new sequence $[\{1/b_n\}]$ is Cauchy. (It is, of course, but we need to show it.) We want to show that given any $\epsilon > 0$ there exists an N such that

$$\frac{1}{b_n} - \frac{1}{b_m} < \epsilon$$

whenever all n, $m > N$. Since we know that $\{b_n\}$ doesn't converge to zero and since we said/showed that the rationals are dense, there has to be a positive number k such that $0 < k < |b_n|$ whenever $n > N$. Moreover Cauchy tell us that we can be sure that $|b_n - b_m| < \epsilon \cdot k^2$ if we choose an even bigger N_1 and n,$m > N_1$. So if n, $m > N_1$

$$|b_n - b_m| < \epsilon \cdot k^2 < \epsilon \cdot |b_n| \cdot |b_m|.$$

Divide through and we're left with

$$\frac{1}{b_n} - \frac{1}{b_m} = \frac{|b_n - b_m|}{|b_n| \cdot |b_m|} < \epsilon.$$

Now we can say with confidence that if $x \in \mathbb{R} - \{0\}$ then $1/x \in \mathbb{R}$, too.