Solution Set 3B

Math 23b
February 26, 2003

4. b) If $(x, y) \neq (0,0)$ then f is a well-defined function given by

$$f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2}$$

in a neighborhood of (x, y), so we can use the product and quotient rule to differentiate, as usual. We have

$$(D_1 f)(0,y) = 0 \cdot D_1\left(\frac{x^2 - y^2}{x^2 + y^2}\right) + y\frac{-y^2}{y^2} = -y$$
$$(D_2 f)(x,0) = 0 \cdot D_2\left(\frac{x^2 - y^2}{x^2 + y^2}\right) + x\frac{x^2}{x^2} = x.$$

If $x = y = 0$ then we have to resort to first principles to calculate the derivative. We have

$$(D_1 f)(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$
$$(D_2 f)(0,0) = \lim_{h\to 0}\frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

Combining the above two calculations, we have $(D_1 f)(0,y) = -y$ for all $y \in \mathbf{R}$ and $(D_2 f)(x,0) = x$ for all $x \in \mathbf{R}$.

c) By definition

$$(D_2 D_1 f)(0,0) = D_2(-y) = -1$$
$$(D_1 D_2 f)(0,0) = D_1(x) = 1$$

which are clearly not equal. One should check this definition to make sure that the derivative $D_2(D_1 f)(0,0)$ really only depends on the function $(D_1 f)(0,y)$.

Note on this problem:

(1) The only tricky part in this problem was realizing that one has to calculate the first derivatives at $(0,0)$ *as a special case*, from first principles. It is true that formally, one can calculate $D_1 f(0,y)$ to be $-y$ at all points, but that calculation is not valid at $y = 0$ for the simple reason that the formula $xy(x^2 - y^2)(x^2 + y^2)$ is not defined at $(0,0)$. If you evaluate the limit explicitly, you should notice that you have to take the limit *differently* in the case $y = 0$ and $y \neq 0$, so it really is a special case. Another way of putting this is that it's really only luck that allows $D_1 f(0,y)$ to be $-y$ at 0; one can come up with similar examples where this is not true.

Joe Rabinoff