Math 23b, Spring 2003

Problem Set 3, Part E
Solutions written by Tseno Tselkov

Problem 7: *Consider the function $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(x) = ||x||x$. Determine whether or not f is differentiable at 0. If not, why not? If so, find the first-order partial derivatives of f at 0. (Bonus: Do the second-order partial derivatives of f exist at 0?)*

Proof. f is differentiable at 0 if and only if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ such that $\lim_{||h|| \to 0} \frac{f(h) - f(0) - L(h)}{||h||}$ exists. Let $L(x) = 0$ for all $x \in \mathbb{R}^n$. L is certainly a linear map and we have

$$\lim_{||h|| \to 0} \frac{f(h) - f(0) - L(h)}{||h||} = \lim_{||h|| \to 0} \frac{||h||h}{||h||} = \lim_{||h|| \to 0} h = 0.$$

The existence of this limit implies that f is differentiable at 0. Moreover, the Jacobian L of f at 0 is the zero matrix, so all first-order partial derivatives at 0 are 0.

We can also deal with this part directly:

$$\frac{\partial f_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(x_i\sqrt{x_1^2 + \ldots + x_i^2 + \ldots + x_n^2}\right) =$$

$$= \sqrt{x_1^2 + \ldots + x_i^2 + \ldots + x_n^2} + \frac{x_i^2}{\sqrt{x_1^2 + \ldots + x_i^2 + \ldots + x_n^2}} = ||x|| + \frac{x_i^2}{||x||}.$$

And also,

$$\frac{\partial f_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(x_j\sqrt{x_1^2 + \ldots + x_i^2 + \ldots + x_n^2}\right) =$$

$$= x_j \frac{1}{2\sqrt{x_1^2 + \ldots + x_i^2 + \ldots + x_n^2}} 2x_i = \frac{x_j x_i}{||x||}.$$

Now let's consider

$$\lim_{||x|| \to 0} \frac{\partial f_i}{\partial x_i} = \lim_{||x|| \to 0} ||x|| + \lim_{||x|| \to 0} \frac{x_i^2}{||x||} = 0 + \lim_{||x|| \to 0} \frac{x_i^2}{||x||}.$$

But $\frac{x_i^2}{||x||} \leq \frac{x_i^2}{\sqrt{x_i^2}} = \frac{x_i^2}{|x_i|}$ and $\lim_{||x|| \to 0} \frac{x_i^2}{|x_i|} = 0$ (clearly) $\Rightarrow \lim_{||x|| \to 0} \frac{\partial f_i}{\partial x_i} = 0$. And similarly for $\lim_{||x|| \to 0} \frac{\partial f_j}{\partial x_i} = 0$. Thus, all first-order partial derivatives exist and are continuous, and are 0 at 0, so we're done with that part.

Let's now deal with the bonus part. The answer is No and the reason is the following: Look even at the simplest case when $n = 1$. Then $f(x) = |x|x \ \Rightarrow f'(x) = |2x|$, which is well-known to be not differentiable at 0. Similar phenomenon can be also observed in the general case. $\square$