Math 23a, 2002.
Solution Set 2, Questions 5-6.
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Question 9. Let V be a vector space over the field F defined as all infinite sequences of elements of F :

$$V = \{(a_0, a_1, a_2, \dots)|a_i \in F, \forall i \geq 0\}$$

For $k = 1, 2$ we definte the two subspaces:

$$\ell^k = \left\{(a_0, a_1, a_2, \dots)|a_i \in F, \forall i \geq 0, \text{ and } \sum_{n=0}^{\infty} |a_n|^k < \infty\right\}$$

(a) (*Not required*) Show/convince yourself that V is a vector space.
(b) Show that ℓ^2 is a subspace of V by showing that addition of vectors and scalar multiplication are closed operations. That is, if $\mathbf{u}, \mathbf{v} \in \ell^2$ and $a \in \mathbb{R}$, then $\mathbf{u} + \mathbf{v} \in \ell^2$ and $a\mathbf{u} \in \ell^2$.
(c) Show that if $F = \mathbb{R}$, then $\ell^1 \subset \ell^2$, and that $\ell^1 \neq= \ell^2$.
(d) (*Bonus*) Consider the spaces ℓ^1 and ℓ^2 defined above with the terms a_i taken from the finite field $F = \mathbb{Z}/2\mathbb{Z}$. Give an explicit condition for deciding whether or not a particular sequence is in ℓ^1 or ℓ^2. (What must happen for such an infinite series to converge? Recall that the limits and sums are being taken in the finite field!)

Answer. I think most of you convinced yourself of part (a). Some of you even showed it in your problem sets. But next time if you have an optional part, and it's long and not for credit, feel free not to write it up. :]

For (b) it's good to start out with a few properties of the binomial square: $(v+u)^2 = u^2 + 2uv + v^2$ and since $(u - v)^2 \geq 0,\ u^2 - 2uv + v^2 \geq 0,$ giving $u^2 + v^2 \geq 2uv$. Now let's get to work. If $v,\ u \in \ell^2$, then both $\sum_{n=0}^{\infty} |v_n|^2$ and $\sum_{n=0}^{\infty} |u_n|^2$ converge. To say that the sum of their series also converges isn't enough. We need to show that vector-sum's series converges. So here's how it goes:

$$\begin{aligned}
\sum_{n=0}^{\infty} |v_n + u_n|^2 &\leq \sum_{n=0}^{\infty} v_n^2 + 2|u_n v_n| + u_n^2 \text{ (by the triangle inequality)} \\
&\leq \sum_{n=0}^{\infty} 2(v_n^2 + u_n^2) \text{ (by what we did above)} \\
&= 2\sum_{n=0}^{\infty} v_n^2 + 2\sum_{n=0}^{\infty} u_n^2 \text{ (since you can factor out the 2)}
\end{aligned}$$

And that last guy converges since $v,\ u \in \ell^2$. So one down, one to go. If $v \in \ell^2$, we need to show $av \in \ell^2$, too. We have

$$\sum_{n=0}^{\infty} |av_n|^2 = \sum_{n=0}^{\infty} a^2 v_n^2 = a^2 \sum_{n=0}^{\infty} v_n^2.$$

No one would argue that that series does anything but converge. So when you're walking down the street and someone asks you, "Is little-ell-two a subspace?" you can smile and assure them that it is.

But we're not done. In part (c) we need to show that if $v \in \ell^1$, that same v lives in ℓ^2, i.e., $\ell^1 \subset \ell^2$. We're given that $\sum_{n=0}^{\infty} |v_n|$ converges. Equipped with that, our task is to that $\sum_{n=0}^{\infty} v_n^2$ converges as well. First remember that if a series is going to converge, its terms must go to zero. More precisely, for all $\epsilon > 0$ there exists an $N > 0$ such $|v_n| < \epsilon$ whenever $n > N$. Surely, after some point all the terms are less than 1 (just choose $\epsilon = 1$, for instance.) Then we have $v_n^2 < |v_n|$ for all $n > N$ and by the comparison test, $\sum_{n=N+1}^{\infty} v_n^2 < \sum_{n=N+1}^{\infty} |v_n| < \infty$. And those beginning terms that were bigger than one? Well, there were only N of them, so their sum is finite, too.

Of note: even if $v_n \to 0$ as $n \to \infty$, a series might not converge. Everyone used this example to show that $\ell^1 \neq \ell^2$. The harmonic series $\sum_{n=1}^{\infty} 1/n$ divereges, but $\sum_{n=1}^{\infty} 1/n^2$ converges. And oh, when did anyone ever mention the Riemann zeta function? A lot of you invoked it to tell me that the last sum equals $\zeta(2) = \pi^2/6$. I mean sure, but the zeta function is heavy, heavy stuff. Let's keep away from it for now and only mention things in our problem sets that we're willing to prove (unless you can assume it from calculus or something).

And I guess it only fair to tell you how the bonus part worked. Since $x^2 = x$ for all $x \in \mathbb{Z}/2\mathbb{Z}$, $\ell^1 = \ell^2$. So that's good. Now we only need to consider what it means for something to converge in $\mathbb{Z}/2\mathbb{Z}$. No matter what our sum can either be 1 or 0. We usually define convergence of an infinitie sum to be the limit of partial sums and that's what we ought to do here. That means the partial sums must be constant after a certain point; that means that the sequence must contain only zeros after a certain point; that means there can only be a finite amount of ones. I wrote that I gave an extra point for this question, but in reality I recorded two extra points. (Don't worry if you didn't do it. Two points is nothing out of a thousand.)

Question 10. Let $V = \{(a_0, a_1, a_2, \dots)|a_i \in \mathbb{R}\}$ be the vector space of all infinite sequences of real numbers. Let W be the subspace of V consisting of all *arithmetic* sequences. Find a basis for W, and determine the dimension of W. (*A sequence is arthimetic if there is some constant c such that $a_{n+1} - a_n = c$ for all $n \geq 0$.*)

Answer. We can solve for a_n explicitly and find out what all the vectors look like. Well, I'm not going to go through all the steps, but you can probably guess that $a_n = a_0 + cn$ is the correct rule since successive terms differ by that constant c. Then a common vector in W is of the form

$$\begin{aligned}(a_0, a_0 + c, a_0 + 2c, a_0 + 3c, \dots) &= (a_0, a_0, a_0, \dots) + (0, c, 2c, 3c, \dots) \\ &= a_0(1, 1, 1, \dots) + c(0, 1, 2, 3, \dots) \\ &= a_0 v_1 + c v_2.\end{aligned}$$

Now the set $\{v_1, v_2\}$ span W by construction. If they're linearly independent they'll be a basis for W and then $\dim(W) = 2$. So let's check that. For $c_1 v_1 + c_2 v_2$ to equal zero, $c_1 \cdot 0 + c_2 \cdot 1 = 0$ in the first component of the sum. Well, that only happens when $c_1 = 0$. And for the rest of the compents to be zero, c_2 better be zero, too. We've found a basis, proved that it's a basis and counted them (giving the *finite* dimension of a vector space of *infinite* sequences). *Neat!*