MATH 23a, FALL 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment #4
Due: October 17, 2003

1. Read Sections 5.1–5.2 of Schneider and Barker and Sections 1.4–1.5 of Edwards.

2. (*) Let $L : V \to W$ be a linear map. Show that L is injective if and only if $Ker(L) = \{\mathbf{0}\}$.

3. (A) Let $P_n(\mathbb{R}) = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}, \forall i\}$ be the vector space of all polynomials of degree less than or equal to n. Consider the map $L : P_n(\mathbb{R}) \to \mathbb{R}$ defined by $L(p) = \int_0^1 p(x)\, dx$.

   (a) Show that L is a linear map.

   (b) Determine $Im(L)$, and find a basis.

4. (A) In this problem, we consider the *shift* operator. Consider the linear map $S : \mathbb{R}^3 \to \mathbb{R}^3$ which acts as follows:

$$S(x, y, z) = (0, x, y).$$

   Find the kernel and image of S, and verify that

$$dim(Ker(S)) + dim(Im(S)) = dim(\mathbb{R}^3).$$

5. (B) We generalize the notion of the *shift* operator. Let V be the vector space of all infinite sequences of real numbers as in Problem #3.9, and consider the linear maps $S : V \to V$ and $T : V \to V$, where S and T act as follows:

$$S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$$
$$T(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$$

   (a) Find the kernel and image of S. Does the result about the dimensions of kernels and images (the Rank-Nullity Theorem) apply?

   (b) Show that $T \circ S = I$ but that $S \circ T \neq I$, where $I : V \longrightarrow V$ is the identity map.

   (c) Which of S and T is onto? Which is one-to-one? Which is invertible? Explain.

6. (C) Consider the linear differential operator $D : C^\infty \longrightarrow C^\infty$ given by $D(f) = f' + af$, where a is some fixed real number. (Recall that C^∞ is the vector space of all functions which are infinitely-differentiable.)

(a) Find $Ker(D)$.

(b) Show that D is surjective. (Hints: 1. Let $g \in C^\infty$, and show that $f' + af = g$ has a solution. Multiply both sides by the *integrating factor* e^{ax}, and integrate both sides, using the product rule on the left-hand side! 2. Cite an appropriate theorem from Calculus to justify the existence of an anti-derivative.)

7. (D) Let $A : U \to V$ and $B : V \to W$ be linear maps between finite-dimensional vector spaces.

   (a) Show that if A and B are both surjective, then so is $B \circ A$.

   (b) Show that if B is one-to-one, then $Ker(B \circ A) = Ker(A)$.

   (c) Assuming A and B are surjective, show that $dim(Ker(B \circ A)) = dim(Ker(A)) + dim(Ker(B))$.

8. (*) Find the inverse of the linear map $L : (\mathbb{Z}/7\mathbb{Z})^3 \longrightarrow (\mathbb{Z}/7\mathbb{Z})^3$ given by $L(x, y, z) = (x + y + z, 2x + 3y + 4z, 3x + 4y + 6z)$.

9. (E) Let F be any field. Show that the linear operator $L : F^2 \longrightarrow F^2$ given by $L(x, y) = (ax+by, cx+dy)$ is invertible if and only if $ad-bc \neq 0$.

10. (*) Show that if $A : U \longrightarrow V$ and $B : V \longrightarrow W$ are both invertible linear maps, then $(BA)^{-1} = A^{-1}B^{-1}$.