

If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me.

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1. Reading.
2. We are given that $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally invertible, *i.e.*, $f'(a) \neq 0$ for all $a \in \mathbb{R}$. We wish to show that f is injective.

Suppose, on the contrary, that there exist distinct a, b such that $f(a) = f(b)$. Without loss of generality, let $a < b$. Since f is differentiable, we can use the single-variable Mean Value Theorem, which implies that there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. Hence, $0 = f'(c)(b - a)$, which implies either $b = a$ or $f'(c) = 0$, either of which is a contradiction.

3. Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $(x, y) \mapsto (e^x \cos y, e^x \sin y)$. We may compute $Jg(x, y)$:

$$Jg(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}.$$

The determinant is $\det(Jg(x, y)) = e^{2x} \cos^2 y + e^{2x} \sin^2 y = e^{2x}$. $e^{2x} \neq 0$ for all $x \in \mathbb{R}$, so we may conclude, by the Inverse Function Theorem, that g is locally invertible at every point on its domain.

On the other hand, $g(0, 0) = g(0, 2\pi)$, so g is not injective (one-to-one).