Solution Set 4

Math 23a
October 23, 2002

3. By definition, a vector $(x, y, z) \in (\mathbf{Z}/7\mathbf{Z})^3$ is in $\ker L$ if

$$\begin{array}{ccccccc} x & & & + & 2z & = & 0 \\ 2x & + & 3y & + & 4z & = & 0 \\ 4x & + & 3y & + & z & = & 0. \end{array}$$

We write this equation in matrix form:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then ordinary algebraic manipulations of the original three equations become elementary row operations of a matrix (cf. Curtis). We transform our matrix as follows:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 4 & 3 & 1 \end{bmatrix} \begin{matrix} \\ {\scriptstyle R_2 - 2R_1} \\ {\scriptstyle R_3 - 4R_1} \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{matrix} \\ \\ {\scriptstyle R_3 - R_2} \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ {\scriptstyle R_2 \cdot 5} \\ \\ \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore our equations are satisfied if and only if

$$\begin{aligned} x + 2z &= 0 \\ y &= 0. \end{aligned}$$

So since $-2z = 5z$ we can write

$$\ker L = \{(5z, 0, z) \ : \ z \in \mathbf{Z}/7\mathbf{Z}\}.$$

Notes on this problem:

(1) Remember what the definition of division is in an arbitrary field F: for a, b in F we define a/b to be $a \cdot b^{-1}$, where b^{-1} is the element of F such that $b \cdot b^{-1} = b^{-1} \cdot b = 1$. Therefore if you want to divide five by three in $\mathbf{Z}/7\mathbf{Z}$, for instance, you need to find the element a such that $a \cdot 3 = 1$ first (in this case, $a = 5$), and then multiply that by five, so $5/3 = 5 \cdot 5 = 25 = 4$. The *fraction* $5/3 \in \mathbf{Q}$ doesn't make any sense as such in $\mathbf{Z}/7\mathbf{Z}$. You may have lost a point or two if I thought you were confused about this.

Joe Rabinoff