MATH 23A SOLUTION SET #4 (PART D)

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Problem (5). *If V is a finite dimensional vector space and $W \subseteq V$ a subspace of V, prove that $dim(W) \leq dim(V)$. Part of this problem is to show that W has a basis, which can be done by successively choosing appropriate vectors in W.*

Proof. Let $dim(V) = n$. We first convince ourselves that W has a basis, by constructing it using the following procedure:

Step 1. Pick any $w_1 \neq \vec{0}$ in W. Let $S_1 = span(w_1)$. If $S_1 = W$, stop. Else, go to Step 2.

Step i. Pick any $w_i \in W$ such that $w_i \notin S_{i-1} = span(w_1, \ldots, w_{i-1})$. Then, the set $\{w_1, \ldots, w_{i-1}, w_i\}$ is linearly independent. Define the new S_i as $S_i = span(w_1, w_2, \ldots, w_i)$. If $S_i = W$, stop. Else, go to Step $i + 1$.

The above procedure necessarily halts after a finite number of steps. Otherwise, we would have infinitely many linearly independent vectors in W, namely $w_1, w_2 \ldots, w_k, \ldots$, and hence infinitely many linearly independent vectors in V. In particular, we would have $n + 1$ linearly independent vectors $w_1, \ldots, w_{n+1}$ in V. But this is impossible by problem 7 on the previous homework. Thus, after finitely many, say m, steps, we get m linearly independent vectors $w_1, \ldots, w_m \in W$ such that $S_m = span(w_1, \ldots, w_m) = W$. Thus, $\{w_1, \ldots, w_m\}$ is a basis for W and $dim(W) = m$.

Now, suppose $m > n$. Then, $m \geq n + 1$. Consider $w_1, \ldots, w_{n+1}$. From their construction above, these $n + 1$ vectors in V are linearly independent. But, again by problem 7 from previous homework, that is impossible since $dim(V) = n$. Thus, $m \leq n$. □

Notes

Many people used the procedure above to show W has a basis, but didn't mention why that procedure cannot go on forever. Even if you do not recognize yourself as having done so, please reread now once more the part of the proof which shows that $dim(W) < \infty$ to make sure you understand why it has to be so.

Also, note that the trivial vector space, $\{\vec{0}\}$, has dimension 0, not 1, since there is no subset of $\{\vec{0}\}$ which is linearly independent.

Danger! Do not try this at home, or on the midterm!

And, this last and most important danger warning is something that pertains to the majority of you who actually made the following mistake on the homework. Suppose $B = \{v_1, \ldots, v_n\}$ is any basis for V, and construct a basis for W by successively choosing those vectors in B which are in W

until you get a set of vectors which span W. Sounds good. In fact, it sounds too good to be true, and it won't work. Why?

Consider the following example. Let $V = \mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$, and let $v_1 = (1, 1)$, $v_2 = (1, -1)$. If you don't trust me that this is a basis for V, draw a picture. If you still don't trust me, prove it. Convinced?

Now, let $W = \{(x, y)|x, y \in \mathbb{R}, y = 0\} \subseteq V$. If you don't trust me this is a subspace of V, prove it. Convinced?

The trick is, I can't "trim down" the basis $B = \{v_1, v_2\}$ of V to get a basis for W, since neither v_1 nor v_2 are in W. Thus, any procedure that attempts to select those elements of B that span W and proclaim them a basis for W in order to show W has a basis would return the empty set, and we would have to conclude $dim(W) = 0$ which is clearly not true.

To sum up:

It is true that, if $W \subseteq V$ is a subspace of a finite dimensional vector space V, and $B_W = \{w_1, \ldots, w_m\}$ is a basis for W, there exist some $v_{m+1}, \ldots, v_n \in V$ such that $B_V = \{w_1, \ldots, w_m, v_{m+1}, \ldots, v_n\}$ is a basis for V. We say that B_V is obtained by extending the basis B_W of W to V. Such B_V can be constructed by choosing vectors $v_{m+1}, \ldots, v_n$ similarly as in the above procedure. Namely, we would run that procedure on V, but with predetermined choices $v_1 = w_1, v_2 = w_2, \ldots, v_m = w_m$ in the first m steps, so that $S_m = span(B_W) = W$. The procedure would halt at the n-th step, where $n = dim(V)$.

It is not true that, if $W \subseteq V$ is a subspace of a finite dimensional vector space V, and $B_V = \{v_1, \ldots, v_n\}$, there exist some $B_W = \{v_{i_1}, \ldots, v_{i_m}\} \subseteq B_V$ which is a basis for W. It could exist, if we make a lucky choice of basis for V to begin with, mut in general, it will not exist.

What To Remember?

You can extend a basis for a subspace to get a basis for the whole space. You cannot always trim down a basis for the whole space to get a basis for the subspace.