

MATH 23a, FALL 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 5
Due: October 24, 2003

1. To prepare for next week, read Sections 1.1–1.3 and 6.1 of Schneider and Barker.
2. (*) Read Chapter 2 of Schneider and Barker.
3. (*) For more on direct sums, read Halmos, sections 18–19, and Curtis, section 23.
4. (*) For more on quotient spaces, read Halmos, sections 21–22, and Curtis, section 26.
5. (A) Let $C[a, b] = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$. Show that $C[0, 1] \cong C[a, b]$ by constructing an explicit (linear) bijection.
6. (B) Let V be a finite-dimensional vector space, and let $U \subset V$ be a subspace. Show that there exists a subspace $W \subset V$ such that $V = U \oplus W$.
7. (C) Let $P_n(\mathbb{R})$ be the vector space of polynomials of degree less than or equal to n , and define the following subspaces:

$$P_n^0 = \{p(x) \in P_n(\mathbb{R}) \mid p(-x) = p(x), \forall x\}$$

$$P_n^1 = \{p(x) \in P_n(\mathbb{R}) \mid p(-x) = -p(x), \forall x\}$$

(Note that the elements in P_n^0 are known as *even* polynomials, and the elements of P_n^1 are known as *odd* polynomials.)

Show that $P_n(\mathbb{R}) \cong P_n^0 \oplus P_n^1$. (If you use bases for the two subspaces for this argument, then you should show that they *are* bases.)

For problems 8–10, recall the definition of a quotient space from class:

Let U be a vector space over the field F , and let V be a subspace of U . We define the **quotient space** U/V as follows:

$$U/V = \{\mathbf{u} + V \mid \mathbf{u} \in U\}.$$

The elements of the quotient space are called **cosets**, and they have the form

$$\mathbf{u} + V = \{\mathbf{u} + \mathbf{v} \mid \mathbf{v} \in V\}.$$

Note that $\mathbf{u}_1 + V = \mathbf{u}_2 + V$ if and only if $\mathbf{u}_1 - \mathbf{u}_2 \in V$.

If we define addition and scalar multiplication in the natural way (as follows):

$$(\mathbf{u}_1 + V) + (\mathbf{u}_2 + V) = (\mathbf{u}_1 + \mathbf{u}_2) + V$$

$$c \cdot (\mathbf{u} + V) = (c \cdot \mathbf{u}) + V$$

for any $c \in F$ and any $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2 \in U$, then in fact, the quotient space U/V is a vector space over F (see class notes from Friday).

6. (D) Let $P_n(\mathbb{R})$ be the vector space of polynomials (with real coefficients) of degree less than or equal to n , and let $n \geq 3$. Find a basis for $P_n(\mathbb{R})/P_2(\mathbb{R})$.
7. (E) Suppose U is a finite-dimensional vector space and V is a subspace of U . Show that $\dim(U/V) = \dim(U) - \dim(V)$.
8. (deferred) **The Second Isomorphism Theorem**

Suppose V and W are subspaces of a vector space U .

Then $V/(V \cap W) \cong (V + W)/W$.

- (a) Prove the theorem, assuming $\dim(U) < \infty$. (Hint: Construct and extend bases for the various spaces.)
- (b) Prove the theorem without assuming that $\dim(U) < \infty$.
(Of course, if you do this correctly, there is no need to do part (a).)