Math 23b, 2003.
Solution Set 5, Question 1.
Joshua Reyes

Question 1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. Show that f is locally invertible in a neighbourhood of every point except the origin, and compute f^{-1} explicitly.

Answer. Okay, so I think it's easier to demonstrate an inverse and say, "Look, it's got an inverse, so it must be invertible [give or take the domain of the inverse]." Since we have that $x^2 + y^2$, this function begs to be written in polar coords. So compose it with the function

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}.$$

Then we can start re-writing f.

$$f\left(x(r,\theta), y(r,\theta)\right) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) = \left(\frac{r\cos\theta}{r^2}, \frac{r\sin\theta}{r^2}\right) = \left(\frac{\cos\theta}{r}, \frac{\sin\theta}{r}\right).$$

Set that guy equal to something, say $f(x, y) = (\tilde{x}, \tilde{y})$. Then we know that

$$\tilde{x}^2 + \tilde{y}^2 = \frac{1}{r^2}$$

which looks suprisingly familiar. For kicks throw a $\tilde{x}$ or $\tilde{y}$ on top and whoa!

$$\frac{\tilde{x}}{\tilde{x}^2 + \tilde{y}^2} = r\cos\theta = x \text{ and likewise, } \frac{\tilde{y}}{\tilde{x}^2 + \tilde{y}^2} = r\sin\theta = y.$$

So in fact $f \circ f = \text{id}$. This doesn't work at $(0, 0)$ because division by zero isn't cool, but it certainly works everywhere else on a ball of radius $\sqrt{x^2 + y^2}$.