Math 23a, 2002.
Solution Set 5, Question 4.
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Question 4. Consider the linear differntial operator $D : C^\infty \to C^\infty$ given by $D(f) = f' + af$, where a is some fixed real number. (Recall that C^∞ is the vector space of all functions which are infinitely-differentiable.)

(a) Find $\ker(D)$.
(b) Show that D is surjective. (Hint: Let $g \in C^\infty$, and show that $f' + af = g$ has a solution. Multiply both sides by the *integrating factor* e^{ax}, and integrate both sides on the the left using the product rule!)

Answer.

(a) Well, since the kernel is the set of all things that make D zero, we ought to start by finding solutions to $D(f) = 0$. Just since it's easier for me to work in differntial notation, I'll set $f(x) = y$ and use them interchangeably, making f' and dy/dx the same thing. So,
$$\frac{dy}{dx} + ay = 0.$$
Apply a separation of variables trick and integrate:
$$\int \frac{dy}{y} = -a \int dx.$$
Then, $\ln y = -ax + C$ and $y = Ae^{-ax}$ where $A = f(0) \in \mathbb{R}$. So $\ker(D) = \operatorname{span}(e^{-ax})$.

(b) Well, let's use the hint. We need to find f such that $D(f) = g$ for $g \in C^\infty$. But first let's multiply through by e^{ax} to make things nicer and make use of the product rule (backwards).
$$\begin{aligned} e^{ax} g(x) &= e^{ax} \quad \frac{dy}{dx} + ay \\ &= e^{ax} \frac{dy}{dx} + a e^{ax} y \\ &= \frac{d}{dx}(e^{ax} y) \end{aligned}$$
Now you look up the Fundamental Theorem of Calculus because you think it could help you, but you've forgotten its assumptions and conclusions. And it tells you that
$$[e^{ax} f(x)]_{x=0}^{t} = \int_0^t e^{ax} g(x) dx.$$
And then
$$f(t) = e^{-at} \quad f(0) + \int_0^t e^{ax} g(x) dx \quad .$$
I promise if you apply D to such an f you'll get g out. And such an f lives in C^∞ since it's the composition of smooth functions and therefore smooth itself.