

MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 6
Due: March 14, 2003

Homework Assignment #6 (Final Version)

1. Read Fitzpatrick, Chapters 14 (especially 14.2 and 14.3) and 17 (especially 17.3 and 17.4).
2. (A) Find the points on the line $x + y = 10$ and the ellipse $x^2 + 2y^2 = 1$ which are closest.
3. (B) Given x_1, \dots, x_n be positive real numbers, we define their arithmetic and geometric means as follows:

$$A.M. = \frac{x_1 + \dots + x_n}{n}$$

$$G.M. = \sqrt[n]{x_1 \cdots x_n}$$

Use Lagrange multipliers to prove that the geometric mean is always less than or equal to the arithmetic mean by minimizing the function $f(x_1, \dots, x_n) = \frac{1}{n}(x_1 + \dots + x_n)$ on the set $S = \{\mathbf{x} \in \mathbb{R}^n | g(\mathbf{x}) = 0\}$, where $g(x_1, \dots, x_n) = x_1 \cdots x_n - 1$.

4. (C) If a triangle has side lengths x , y , and z and perimeter $2s = x + y + z$, then its area A is given by $A^2 = s(s - x)(s - y)(s - z)$. Show that, among triangles with a given perimeter, the one with the largest area is equilateral. (*Bonus: Prove the area formula!*)
5. (D) Let $f(x, y, z) = xy^2z^3$, and consider the point $\mathbf{a} = (1, 0, -1)$. Find the second-order Taylor polynomial T_2 for f at \mathbf{a} , and show directly that the second-order remainder, defined as $R_2(\mathbf{h}) = f(\mathbf{a} + \mathbf{h}) - T_2(\mathbf{h})$, satisfies:

$$\lim_{\|\mathbf{h}\| \rightarrow 0} \frac{R_2(\mathbf{h})}{\|\mathbf{h}\|^2} = 0$$

Alternatively, if we think of T_2 as a function of (x, y, z) , we can define $R_2(x, y, z) = f(x, y, z) - T_2(x, y, z)$ and show that:

$$\lim_{(x,y,z) \rightarrow (1,0,-1)} \left(\frac{R_2(x, y, z)}{(x-1)^2 + y^2 + (z+1)^2} \right) = 0$$

6. (Not required)

Let $f(x, y) = x^2 \sin y$, and consider the point $\mathbf{a} = (3, \frac{\pi}{2})$. Find the n -th order Taylor polynomials T_n for f at \mathbf{a} when $n = 0, 1, 2, 3$, and express the $T_n(x, y)$ as polynomials in $(x - 3)$ and $(y - \frac{\pi}{2})$. How does T_3 compare with the third-order Taylor polynomials for $g(x) = x^2$ at $a = 3$ and for $h(y) = \sin y$ at $b = \frac{\pi}{2}$? Can you predict T_4 for f at \mathbf{a} in terms of the fourth-order Taylor polynomials for g and h (at a and b , respectively)?

7. (E) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be of class C^3 . Let \mathbf{a} be a critical point of f , that is, $\nabla f(\mathbf{a}) = \mathbf{0}$. Show that if the quadratic form $q(\mathbf{h})$ corresponding to f at \mathbf{a} is positive-definite, then $f(\mathbf{a})$ is a local minimum.

(We are looking for a δ - ε proof using the equation:

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) = q(\mathbf{h}) + R_2(\mathbf{h}).$$

You may use Taylor's Theorem without proof.)