**Math 23b, 2003.**
**Solution Set 6B, Moral Homework.**
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**Question 1.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = x^2 + xy + z^2 - \cos y$.

(a) Show that 0 is a critical point of $f$.
(b) Find the quadratic form $q$ of $f$ at 0, and the associated matrix $A$.
(c) Find the eigenvalues of $A$ and the associated orthonormal eigenbasis for $\mathbb{R}^3$.
(d) Determine the nature of the critical point 0 from a consideration of the eigenvalues of $A$.

**Answer.** Since this guy goes from many to one dimension, let's look at the gradient of $f$

$$\nabla f = \begin{bmatrix} 2x + y \\ x + \sin y \\ 2z \end{bmatrix}.$$

At $x = y = z = 0$, this bad boy is, in fact, zero. Now on to the mixed partials, ho!

$$q(x, y, z) = \vec{x} \cdot A\vec{x} = \frac{1}{2} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 1 & \cos 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

When you expand that out, you'll find that

$$q(x, y, z) = x^2 + xy + \frac{1}{2}y^2 + z^2.$$

To tease out the eigenvalues, figure out the characteristic polynomial of $A$ with some help from our friend the determinant.

$$\begin{aligned} p_A(\lambda) = \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 1/2 & 0 \\ 1/2 & 1/2 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{bmatrix} \\ &= (1 - \lambda) \left( (1 - \lambda) \left( \frac{1}{2} - \lambda \right) - \frac{1}{4} \right) = (1 - \lambda) \left( \lambda^2 - \frac{3}{2}\lambda + \frac{1}{4} \right). \end{aligned}$$

The roots happen at $\lambda_1 = 1$ and at the savoury $\lambda_2$, $\lambda_3 = (3 \pm \sqrt{5})/4$. It's sort of easy to see that the first eigenvector is $v_1 = (0, 0, 1)$. Luckily that'n's already normalised. Use your favourite method [like solving $A\vec{x} = \lambda_i\vec{x}$ or finding $\ker(A - \lambda_i I)$ for $i = 2, 3$ and normalising] to discover the orthonormal eigenbasis

$$v_1 = (0, 0, 1),\ v_2 = \sqrt{\frac{2}{5 - \sqrt{5}}}(1, (\sqrt{5} - 1)/2, 0),\ \sqrt{\frac{2}{5 + \sqrt{5}}}(1, -(\sqrt{5} + 1)/2, 0).$$

Lastly, since all the $\lambda_i > 0$, this fellow's positive definite, telling us that $f(0)$ is a minimum.

**Question 2.** Let $q(x, y, z) = 2x^2 + 5y^2 + 2z^2 + 2xz$ be a quadratic form. Show that $q$ is positive-definite by diagonalising the form.

**Answer.** Luckily we showed that the eigenvalues work independently of matrix representation. So let's pretend $q(\vec{x}) = \vec{x} \cdot A\vec{x}$. Then we'll get something like

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Find $p_A(\lambda)$ as above and factor it in Mathematica or something. You should find that

$$p_A(\lambda) = (5 - \lambda)(3 - \lambda)(1 - \lambda).$$

Since the roots/eigenvalues are all positive, $q$ must work the same way. [Read: positive definite, yo.]

**Question 3.** Find the minimum value of the function $f(x, y, z) = x^3 + y^3 + z^3$ on the intersection of the planes $x + y + z = 2$ and $x + y - z = 3$.

**Answer.** This problem screams for [shudder.] Lagrange multipliers. Call the first plane $g(x, y, z) = x + y + z - 2 = 0$ and the other $h(x, y, z) = x + y - z - 3 = 0$. So let's do it up:

$$\nabla f = (3x^2, 3y^2, 3z^2),\ \nabla g = (1, 1, 1),\ \nabla h = (1, 1, -1).$$

Critical points occur when the gradient of the function is a linear combination of the gradients of the constraints. So I guess you expect me to figure out $\nabla f = \lambda \nabla g + \mu \nabla h$ for you, eh? Oh well, here goes:

$$3x^2 = \lambda + \mu,\ 3y^2 = \lambda + \mu,\ 3z^2 = \lambda - \mu,$$
$$x + y + z = 2 \text{ and } x + y - z = 3.$$

Subtract the bottom two and miraculously $z = -1/2$. Because of the first two $x$ and $y$ are the same up to a sign. But going back and adding those bottom ones, $x + y = 5/2$. So $x = y = 5/4$. So did we find a minimum or a maximum? Pick any other point on the line of intesection, I bet it's more than $f(5/4, 5/4, -1/2) = 121/32$.

**Question 4.** Finish the example from class on Wednesday, March 12. That is, consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$f(x, y) = x + x^2 + xy + y^3.$$

(a) Find all the critical points of $f$ by setting $\nabla f = 0$. (*Hint: There are two.*)
(b) At each critical point of $f$, find the quadratic form, the associated matrix, and its eigenvalues.
(c) Classify the "definiteness" of the quadractic form of $f$ at each critical point, and identify each critical point as a local maximum, minimum, or neither.

**Answer.** Let's do what the man says. Since we're looking when $\nabla f = (1 + 2x + y, x + 3y^2)$ vanishes, we know $x = 3y^2$. Substitute into the first and we've got ourselves the quadratic $6y^2 - y - 1 = (3y + 1)(2y - 1)$. Now we know the critical points $x_1 = (-3/4, 1/2)$ and $x_2 = (-1/3, -1/3)$.

Now it's time to calculate the second-order partials. Luckily they're not too bad.

$$f_{xx} = 2, \quad f_{xy} = f_{yx} = 1, \quad f_{yy} = 6y.$$

So the quadratic forms manifest themselves in their full glory after plugging in.

$$Q_{x_1}(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}^t \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + xy + \frac{3}{2}y^2$$

$$\text{and } Q_{x_2}(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}^t \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + xy - y^2.$$

As for the eigenvalues of $Q_{x_1}$, they happen when $(1 - \lambda)(3/2 - \lambda) - 1/4 = 0$. By inspection, therefore, we see that $(5 \pm \sqrt{5})/4 > 0$ will do. The positivity of these values give us the definite impression that $x_0$ is a minimum.

Likewise, $Q_{x_2}$ has eigenvalues when $(1 - \lambda)(-1 - \lambda) - 1/4 = 0$. Arduous computation shows that $\lambda = \pm\sqrt{5}/2$. Unfortunately, we can't definitely say anything since the form is sometimes positive and sometimes negative. So, $x_2$ is neither a max nor a min. Sorry, guys.