Math 23b, 2003.
Solution Set 6, Question 1.
Joshua Reyes

Question 1. Find the points on the line $x + y = 10$ and the ellipse $x^2 + 2y^2 = 1$ which are closest.

Answer. First off, let's rewrite the line and ellipse so that we don't get variable names confused. While we're at it, let's make these guys level sets.

$$g_1(x, y, s, t) = x + y - 10$$
$$g_2(x, y, s, t) = s^2 + 2t^2.$$

Now that we've got the constraints, let's whip up a function to minimize. Just like in class, I'll go for distance squared, rather than distance. You know,

$$f(x, y, s, t) = (x - s)^2 + (y - t)^2.$$

Lagrange kicks in. So we'll probably need to calculate

$$\nabla g_1 = (1, 1, 0, 0) \text{ and } \nabla g_2 = (0, 0, 2s, 4t).$$

Also, it wouldn't hurt to know that

$$\nabla f = 2(x - s, y - t, s - x, t - y).$$

According to that multiplier theorem, we know that critical points occur when ∇f is a linear combination of the gradients of the contraint functions. In math talk,

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2.$$

Which is really a system of four equations in disguise:

$$\begin{aligned} 2x - 2s &= \lambda_1 & \quad 2y - 2t &= \lambda_1 \\ 2s - 2x &= 2\lambda_2 s & \quad 2t - 2y &= 4\lambda_2 t. \end{aligned}$$

Now type that into your favourite CAS program and hit return. It should tell you that

$$(x, y) = \left(5 \pm (1/2)\sqrt{2/3}, 5 \mp (1/2)\sqrt{1/6}\right) \text{ and } (s, t) = \left(\pm\sqrt{2/3}, \pm\sqrt{1/6}\right).$$

Then if you plug each combination in by hand, you should pick out

$$(x, y) = \left(5 + (1/2)\sqrt{2/3}, 5 - (1/2)\sqrt{1/6}\right) \text{ and } (s, t) = \left(\sqrt{2/3}, \sqrt{1/6}\right).$$