Math 23b, Spring 2003

Problem Set 6, Part C
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Problem 4: *If a triangle has side lengths $x, y,$ and z and perimeter $2s = x+y+z$, then its area is given by $A^2 = s(s-x)(s-y)(s-z)$. Show that among the triangles with a given perimeter the one with greatest area is the equilateral. (Bonus: Prove the area formula!)*

Proof. We want to maximize $f(x, y, z) = s(s-x)(s-y)(s-z)$ subject to $g(x, y, z) = x+y+z-2s = 0$. Let $S \subset \mathbb{R}^3$ be such that $x+y+z = 2s$ $(x, y, z \geq 0)$. We allow zero lengths and we don't care about the triangle equality (thus we'll obtain a maximum on a larger set than the allowed one and if this maximum satisfies the triangle inequality we'll be done). This S is clearly closed and bounded and consequently compact. Thus we can apply Lagrange multipliers to maximize f on S subject to $g = 0$. Looking for a maximum inside S we get that for some λ, $\nabla f = \lambda \nabla g$, i.e. $(-s(s-y)(s-z), -s(s-x)(s-z), -s(s-x)(s-y)) = (\lambda, \lambda, \lambda)$. This easily gives us the extremum $x = y = z$, which is easily checked to be a maximum. However, to prove that from the triangles with perimeter $2s$, the one with maximal area is the equilateral, we do need to check what happens on the boundary. And the boundary of S is given by $x = 0$, or $y = 0$, or $z = 0$. When $x = 0$, $y + z = 2s$ and $A^2 = s(s-x)(s-y)(s-z) = ss(s-y)(y-s) \leq 0$. Thus the maximal area occurs when $x = y = z$, which is a valid triangle and we are done with this problem.

Bonus: The area is $A^2 = x^2y^2 \sin^2 \gamma$. By the Cosine Law we have $\cos^2 \gamma = \left(\dfrac{x^2+y^2-z^2}{2xy}\right)^2$. Consequently,

$$A^2 = x^2y^2 \left(1 - \left(\frac{x^2+y^2-z^2}{2xy}\right)^2\right) = s(s-x)(s-y)(s-z).$$

Notes: There are two important things to do when you use Lagrange multipliers:

- Consider your function on a **compact** set;
- Always check what happens on the **boundary**.

You know that on a compact set the maximum/minimum occurs either inside the set (and you can obtain those using the Lagrange multiplier) or on the boundary (which you check manually). □