Math 23a, 2002.
Solution Set 6, Questions 6 and 7.
Joshua Reyes

Question 6. Suppose $A : V \to V$ is a linear map, and suppose $\{v_1, \ldots, v_m\}$ is a set of non-zero eigenvectors for A with distinct eigenvalues $\lambda_1, dots, \lambda_m$. Show that these vectors are linearly independent. (Hint: Use induction on m.)

Answer. Let's use the hint and applying induction on m. So start off with the dependence relation

$$c_1 v_1 + \cdots + c_m v_m = \sum_{i=1}^{m} c_i v_i = 0$$

for appropriate $c_i \in F$. As we're trying to prove linear independence, we hope that those c_i are secretly all zero. But for now, let's assume that at least two of the c_i are non-zero. (Notice that if all but one, say c_j, were zero, then v_j would have to be zero, too. But eigenvectors can't be zero by definition.) Since we've got no where else to turn, we apply A remembering to exploit linearity:

$$A \left. \sum_{i=0}^{m} c_i v_i \right) = \sum_{i=1}^{m} c_i A(v_i) = \sum_{i=1}^{m} c_i \lambda_i v_1 = A(0) = 0.$$

Now we have two equations in the v_i. A reasonalbe thing to do would be to eliminate one of them, say v_m. So multiply our first relation through by λ_m and subtract. We're left with

$$c_1(\lambda_1 - \lambda_m)v_1 + \cdots + c_{m-1}(\lambda_{m-1} - \lambda_m)v_{m-1} = 0.$$

Now comes the inductive step. By our assumptions $\lambda_i - \lambda_m \neq 0$ for $i \neq= m$. Also, at least one c_i is non-zero, leaving $v_1, \ldots, v_{m-1}$ linearly dependent. Rinse and repeat the above procedure to conclude that $v_1, \ldots, v_{m-2}$ are also linearly dependent. In fact, do this until v_1 is linear dependent, i.e., until $v_1 = 0$. But wait! That's a dirty lie. We know that eigenvectors are always non-zero. Moral of the story: the c_i were all zero to begin with and the v_i had to be linearly independent.

Question 7. Let $V = (\mathbb{Z}/7\mathbb{Z})^3$, and consider the linear map $L : V \to V$ given by $L(x, y, z) = (x + y + z, 2y + 3z, 4z)$. Find the eigenvalues of L, and find an eigenbasis for V. (Hint: Look for likely choices of eigenvalues—I claim that two of them are easy, and the third follows a pattern.)

Answer. Since I can never visualise these things, let's slap the coefficients into a matrix (as per usual). Then the matrix of L with respect to the standard basis vectors acting on a vector $x \in V$ looks like

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix} = \begin{bmatrix} x + y + z \\ 2y + 3z \\ 4z \end{bmatrix}.$$

We can see from the matrix that $L(e_1) = 1e_1$. Eigenvalue $\lambda_1 = 1$ with vector $v_1 = e_1$. Unfortunately, as is usually the case, the other two aren't so obvious. Subtracting the two rightmost terms in the three-part equality and reconstructing our matrix we see that

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

Alternatively, $v_\lambda \in \ker(A - \lambda I)$. We already know what happens when $\lambda = 1$. Smart guessing suggests we try $\lambda = 2, 4$. Then thanks to Gauss-Jordan elimination, we find that when $\lambda = 2$, $v_2 = (1, 1, 0)^t$ and when $\lambda = 4$, $v_4 = (1, 6, 4)^t$. By question 6, we know that these three are linearly independent. And since we have three linearly independent vectors in a 3-dimensional space, they've got to span. Tada.