MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 7
Due: April 11, 2003

Homework Assignment #7 (Final Version)

1. Read Fitzpatrick, Sections 18.1–18.2.

2. (A) In class, we proved the following theorem:

   Theorem. Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \longrightarrow \mathbb{R}$ be a bounded function. If f is continuous at a, then $o(f, a) = 0$.

   Prove the converse. (Recall that $o(f, a)$ is the *oscillation* of f at a.)

3. (A) Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \longrightarrow \mathbb{R}$ be a bounded function. If P and P' are any two partitions of A, show that

$$L(f, P) \leq U(f, P').$$

4. (B) Let $f : A \longrightarrow \mathbb{R}$ be integrable. Show that $|f|$ is integrable and that

$$\left| \int_A f \right| \leq \int_A |f|.$$

5. (B) Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let

$$I = \{f : A \longrightarrow \mathbb{R} | f \text{ is integrable on } A\}.$$

   (a) Show that I is a vector space over $\mathbb{R}$ by showing that if $f_1, f_2 \in I$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2 \in I$. (In other words, I is a subspace of the vector space $V = \{f : A \longrightarrow \mathbb{R}\}$.)

   (b) Show that $\int_A cf = c \int_A f$ and $\int_A (f_1 + f_2) = \int_A f_1 + \int_A f_2$.

6. (C) Let $A = [0, 1] \times [0, 1]$, and define $f : A \longrightarrow \mathbb{R}$ as follows:

$$f(x, y) = \begin{cases} 0 & , \text{ if } x \notin \mathbb{Q} \\ 0 & , \text{ if } x \in \mathbb{Q} \text{ and } y \notin \mathbb{Q} \\ \frac{1}{q} & , \text{ if } x \in \mathbb{Q} \text{ and } y = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

   (a) For each $\mathbf{a} \in A$, determine $o(f, \mathbf{a})$.

   (b) Show that f is integrable on A. (What is $\int_A f$?)

7. (C) Give an example of a closed set of measure zero that does not have content zero.

8. (D) Let $A = \{x \in [0,1] | \text{the decimal expansion of } x \text{ has no 8's}\}$.

   Let $B = \{n \in \mathbb{N} | \text{the decimal expansion of } n \text{ has no 8's}\}$.

   (a) Find the content of A.

   (b) Decide whether the infinite series $\displaystyle\sum_{n \in B} \frac{1}{n}$ converges or diverges.

9. (D) For a function $f : [0,1] \longrightarrow \mathbb{R}$, let $A = \{x \in [0,1] | f \text{ is not differentiable at } x\}$. Find such an f satisfying the following conditions:

   - f is continuous.
   - $f(0) = 0$
   - $f(1) = 1$
   - A has content zero.
   - If $x \notin A$, then $f'(x) = 0$.

   (Hint: Use the Cantor set.)