

Math 23b, Spring 2003

Problem Set 7, Part C
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Problem 6: Let $A = [0, 1] \times [0, 1]$, and let $f : A \rightarrow \mathbb{R}$ send (x, y) to $\frac{1}{q}$ if $x, y \in \mathbb{Q}$ and $y = \frac{p}{q}$ in lowest terms, and to 0 otherwise.

(a) For each $a \in A$, determine $o(f, a)$.

(b) Show that f is integrable on A and find $\int_A f$.

Proof. (a) Recall the definition of oscillation at a point a :

$$o(f, a) = \lim_{\delta \rightarrow 0} [M(f, a, \delta) - m(f, a, \delta)].$$

By the density of the irrational numbers we have that in any neighborhood of a there is a point with irrational y coordinate, which implies that $m(f, a, \delta) = 0$, and consequently $o(f, a) = \lim_{\delta \rightarrow 0} M(f, a, \delta)$. We now need to consider two separate cases.

Case 1: $a = (x, y)$ with $y \in \mathbb{Q}, y = \frac{p}{q}$ in lowest terms. We claim that in this case $o(f, a) = \frac{1}{q}$. Clearly $M(f, a, \delta) \geq \frac{1}{q}$ since in every neighborhood of (x, y) we can certainly find a point $b = (x', y)$ with $x' \in \mathbb{Q}$ and clearly $f(b) = \frac{1}{q}$. Now we claim that in fact $\lim_{\delta \rightarrow 0} M(f, a, \delta) = \frac{1}{q}$. Indeed, there are finitely many $n \in \mathbb{N}, n < q$ and let us denote by δ_n the smallest distance of a point of the form $\frac{k}{n}$ to $\frac{p}{q}$ ($k \in \mathbb{N}$). Let $\delta < \min(\delta_1, \dots, \delta_{q-1})$. By construction in the ball $B_\delta(a)$ there is no rational point with denominator of the y coordinate less than q , which implies that $M(f, a, \delta) \leq \frac{1}{q}$ for all those sufficiently small δ , which verifies our assertion.

Case 2: $a = (x, y)$ with $y \notin \mathbb{Q}$. In this situation an identical argument to the one above shows that given any $q \in \mathbb{N}$, for all δ sufficiently small $M(f, a, \delta) \leq \frac{1}{q}$, which directly implies that $\lim_{\delta \rightarrow 0} M(f, a, \delta) = 0$,

i.e. $o(f, a) = 0$.

Summing up, $o(f, a) = 0$ if $y \notin \mathbb{Q}$, and $o(f, a) = \frac{1}{q}$ if $y = \frac{p}{q}$ in lowest terms.

(b) Recall that a function is integrable if the set of its discontinuities has measure 0, which is the set of points where the oscillation is more than 0. By (a) this is the set $\mathbb{R} \times \mathbb{Q} \cap [0, 1] \times [0, 1]$. But using the fact that \mathbb{Q} is countable we can cover that set by rectangles of the form $[0, 1] \times \left[q_i - \frac{\epsilon}{2^{i+2}}, q_i + \frac{\epsilon}{2^{i+2}} \right]$, which have total volume $< \epsilon$, i.e. the set of discontinuities is of measure 0, i.e. f is integrable on A . But clearly the $\sup L(f, P) = 0$, which implies $\int_A f = 0$. \square

Problem 7: Give an example of a closed set of measure zero that does not have content zero.

Proof. Let's consider $\mathbb{Z} \subset \mathbb{R}$. It is closed since its complement clearly is a union of open intervals. As we showed in class it has measure zero, but undefined content. \square