MATH 23a, FALL 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 8
Due: December 5, 2003

1. Read Chapters 4 (especially sections 4.1–4.3) and 7 (especially sections 7.1–7.3) of Schneider and Barker and Sections 1.3 and 1.6 of Edwards.

2. (A) Recall the definition of the *transpose* of a matrix, as referred to in homework problem #8.8, and prove the following:

   Theorem: If A is an $n \times n$ matrix, then $det(A^t) = det(A)$.

3. (B) Let $A : V \longrightarrow V$ be a linear transformation on a finite-dimensional vector space, and by slight abuse of notation, let A also be the matrix for this transformation with respect to a fixed basis. Using the following method, we determine the eigenvalues of A:

$$\begin{aligned} \lambda \text{ is an eigenvalue for } A &\iff V_\lambda = Ker(A - \lambda I) \text{ is non-trivial} \\ &\iff A - \lambda I \text{ is not invertible} \\ &\iff det(A - \lambda I) = 0 \end{aligned}$$

   Thus we are inspired to make the following definition:

   $p_A(\lambda) = det(A - \lambda I)$ is called the **characteristic polynomial** of A

   The eigenvalues of A will be the roots of the characteristic polynomial.

   (a) Prove that no scalar $\lambda_0 \in F$ is an eigenvalue for A unless it is a root of $p_A(\lambda)$.

   (b) If $p_A(\lambda) = (\lambda - \lambda_0)^k \cdot q(\lambda)$ with $q(\lambda_0) \neq 0$, then we say that the eigenvalue λ_0 has *algebraic multiplicity* equal to k. (That is, λ_0 is a root of $p_A(\lambda)$ of order k.) Show that the geometric multiplicity (which, by definition, is the dimension of the corresponding eigenspace) of an eigenvalue is less than or equal to its algebraic multiplicity.

   (c) Use this method to find all eigenvalues of the real matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

   Is the matrix A diagonalizable? Explain.

4. (A) Show that $A : V \longrightarrow V$ is invertible if and only if $det(A) \neq 0$. (We have used this fact several times already, including in problem #3. The point of this exercise is to make you think carefully about the steps we used when we made the transition from alternating forms to determinants.)

5. (*) Write down a real number that: 1. you have never seen written down before, and 2. does not have a finite algebraic representation in terms of standard mathematical symbols. (For example, $\pi + 1$ is certainly a real number, but it does not satisfy condition 2.)

6. (C) In this problem, you will show that every real number (with a few exceptions!) has a unique decimal expansion.

   Let $x > 0$ be a real number. Show that there is an integer k and integers $a_i \in \{0, 1, 2, \ldots, 9\}$ for every $i \geq k$ such that x may be represented in the form:

$$\begin{aligned} x &= \sum_{i=k}^{\infty} a_i \cdot 10^{-i} \\ &= a_k \cdot 10^{-k} + a_{k+1} \cdot 10^{-k-1} + \cdots + a_{-1} \cdot 10^{1} + a_0 \cdot 10^{0} + a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + \cdots \end{aligned}$$

   Show that this representation is unique, except in the case where there exists some $n \in \mathbb{N}$ such that $10^n \cdot x \in \mathbb{N}$.

   (Hint: Use the Well-Ordering Principle and perhaps the Division Algorithm.)

7. (*) Prove that $\sqrt[3]{2}$ is an irrational real number. (In other words, show that there is a real number x that satisfies the equation $x^3 - 2 = 0$, but that x is not rational.)

8. (D) Considering the real numbers as defined by equivalence classes of Cauchy sequences of rational numbers, name the equivalence class that acts as the multiplicative identity, and verify that it does.

9. (D) Considering the real numbers as defined by equivalence classes of Cauchy sequences of rational numbers, prove the existence of multiplicative inverses (for elements other than the additive identity).

10. (E) Use the definition of Cauchy sequence to show that the sequence of rational numbers $\left\{ \frac{1}{n^2} \right\}_{n=1}^{\infty}$ is a Cauchy sequence.

11. (deferred) Let a and b be real numbers. Using P to denote the set of positive elements, we define "less than" formally by the statement:

$$a < b \quad \text{iff} \quad b - a \in P.$$

   Use axioms P1–P3 for an ordered field and the definition of absolute value to show that:

   - If $0 < a < b$, then $0 < a^2 < b^2$.
   - If $a^2 < b^2$, then $|a| < |b|$.

12. (deferred) Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$ and an associated norm $|| \cdot ||$. We say that a linear transformation $A : V \longrightarrow V$ is *norm-preserving* if $||A\mathbf{v}|| = ||\mathbf{v}||$ for every $\mathbf{v} \in V$ and *inner-product-preserving* if $\langle A\mathbf{u}, A\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$.

   (a) Show that A is inner-product-preserving if and only if it is norm preserving. (Hint #1: One way is easy! Hint #2: Expand the identity $||A(\mathbf{u} + \mathbf{v})||^2 = ||\mathbf{u} + \mathbf{v}||^2$.)

   (b) Let $V = \mathbb{R}^2$ with the usual inner product. Find all norm-preserving linear transformations/matrices.

13. (deferred) Orthogonalizing a set of functions:

   (a) Consider the vector space $V = C[-1, 1]$ of real-valued continuous functions on the closed interval $[-1, 1]$ with inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

   Orthogonalize the set of functions $\{1, x, x^2, x^3\}$ with respect to this inner product.

   (b) With $V = C[0, 1]$ and inner product

$$\langle f(x), g(x) \rangle = \int_{0}^{1} f(x)g(x)\, dx,$$

   orthogonalize the same set of functions from part (a).

   (c) Now let $V = C[-1, 1]$ as in part (a), but define $\langle \cdot, \cdot \rangle$ as in part (b). Show that this bilinear form is *not* an inner product on V. (Which properties of an inner product *are* satisfied?)