MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 8
Due: April 18, 2003

Homework Assignment #8 (Final Version)

1. Read Fitzpatrick Sections 18.3 and 18.4.

2. (A) In class, we proved a the following theorem:

   **Theorem.** Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \longrightarrow \mathbb{R}$. Let $B = \{x \in A | f \text{ is discontinuous at } x\}$. Then $f$ is integrable if and only if $B$ has measure zero.

   In the "if" part of the proof, we considered the set:

   $$B^\varepsilon = \{x \in A | o(f, x) \geq \varepsilon\}$$

   and claimed that it was closed (and hence compact since $A$ is bounded). Prove this.

3. (B) Let $f : A \longrightarrow \mathbb{R}$ and $g : A \longrightarrow \mathbb{R}$ be two bounded functions such that the closure of $B = \{x \in A | f(x) \neq g(x)\}$ is a set of measure zero. (This implies that $f$ and $g$ agree except on a set of measure zero.) Show that $f$ is integrable if and only if $g$ is, and that if they are both integrable, then $\displaystyle\int_A f = \int_A g$.

4. (C) In class, we considered the set $A = \mathbb{Q} \cap [0, 1]$ and showed that it has measure zero. In particular, we showed that it was countable, that is, we could write it as $A = \{a_0, a_1, a_2, a_3 \ldots\}$. Given an $\varepsilon > 0$, we then covered it with rectangles $I_i = [a_i - \frac{\varepsilon}{2^{i+2}}, a_i + \frac{\varepsilon}{2^{i+2}}], \forall i \in \mathbb{N}$, so that

   $$v\left(\bigcup_{i=0}^{\infty} I_i\right) \leq \sum_{i=0}^{\infty} v(I_i) = \varepsilon.$$

   Fix $\varepsilon = \frac{1}{2}$, and let $J_i = (a_i - \frac{1}{2^{i+3}}, a_i + \frac{1}{2^{i+3}})$ be the open rectangle equal to the interior of the corresponding $I_i$ defined above.

   Let $B = \bigcup_{i=2}^{\infty} J_i$. *(Note that we are purposely omitting the first two sets, which cover 0 and 1, respectively, so that each $J_i \subset [0, 1]$!)*

   (a) Show that $\partial B = [0, 1] \setminus B$.

   (b) Show that $\partial B$ does not have measure zero.

(c) Let $\chi_B$ be the characteristic function of $B$. Show that $\chi_B$ is not integrable on $[0, 1]$.

*(Note that although $B$ is a "reasonable" set in the sense that it is the union of a countable collection of open sets, it does not have a "reasonable" boundary, and so $\chi_B$ is not integrable.)*

5. (D) Let $S \subset \mathbb{R}^3$ be the (bounded) intersection of the two (unbounded) cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$. Show that the volume of $S$ is $\frac{16}{3}$.

   You can view an interactive feature with this object at:

   `http://www.math.umn.edu/~garrett/qy/Cylinders.html`

6. (E) Show that the volume of $B_1(0) \subset \mathbb{R}^4$ is $\frac{\pi^2}{2}$.

   *(Hint: Use the fact that the volume of a ball in $\mathbb{R}^3$ of radius $r$ is $\frac{4}{3}\pi r^3$.)*