Math 23b, Spring 2003

Problem Set 8, Part A
Solutions written by Tseno Tselkov

Problem 2: *Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \to \mathbb{R}$ be a bounded function. Prove that the set*

$$B^\epsilon = \{x \in A | o(f, x) \geq \epsilon\}$$

is closed.

Proof. We need to show that $\mathbb{R}^n - B^\epsilon$ is open. If $x \in \mathbb{R}^n - B^\epsilon$, then either $x \notin A$ or $x \in A$ and $o(f, x) < \epsilon$.

In the first case, since $\mathbb{R}^n - A$ is open, there is an open rectangle C containing x such that $C \subset \mathbb{R}^n - A \subset \mathbb{R}^n - B^\epsilon$.

In the second case by the definition of oscillation there is a $\delta > 0$ such that $M(x, f, \delta) - m(x, f, \delta) < \epsilon$. Let C be an open rectangle containing x such that $|x - y| < \delta$ for all $y \in C$. We will be done if we show that $C \subset \mathbb{R}^n - B^\epsilon$. To do that it suffices to show that if $y \in C$ there is a δ_1 such that $M(y, f, \delta_1) - m(y, f, \delta_1) < \epsilon$. But if $y \in C$ there is δ_1 such that $|x - z| < \delta$ for all z satisfying $|z - y| < \delta_1$ and then $M(y, f, \delta_1) - m(y, f, \delta_1) < \epsilon$ follows directly from $M(x, f, \delta) - m(x, f, \delta) < \epsilon$. □