Solution Set 8E

Math 23b
May 2, 2003

6. We write

$$B_1(0) = \{(x, y, z, w) \in \mathbf{R}^4 \ : \ x^2 + y^2 + z^2 + w^2 \le 1\}.$$

Using this notation and splitting into iterated integrals, the volume of $B_1(0)$ is

$$\int_{B_1(0)} dx\, dy\, dz\, dw. \tag{1}$$

If w is fixed between -1 and 1, then the set of (x, y, z) that we will integrate over in the above integral is given by

$$S_r = \{(x, y, z) \in \mathbf{R}^3 \ : \ x^2 + y^2 + z^2 \le 1 - w^2\}$$

which is the three-sphere of radius $\sqrt{1 - w^2}$. We know that the volume of this sphere is $4\pi(1 - w^2)^{3/2}/3$, so (1) is given by

$$\frac{4}{3}\pi \int_{-1}^{1} (1 - w^2)^{3/2}\, dw.$$

In order to do this integral, we make the substitution $w = \sin\theta$, so $dw = \cos\theta\, d\theta$ and our new bounds of integration are $-\pi/2$ to $\pi/2$. Our integral is thus

$$\begin{aligned}
\frac{4}{3}\pi \int_{-\pi/2}^{\pi/2} (1 - \sin^2\theta)^{3/2} \cos\theta\, d\theta &= \frac{4}{3}\pi \int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta \\
&= \frac{4}{3}\pi \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}(\cos 2\theta + 1)\right)^2 d\theta \\
&= \frac{1}{3}\pi \int_{-\pi/2}^{\pi/2} (\cos^2 2\theta + 2\cos 2\theta + 1)\, d\theta \\
&= \frac{1}{3}\pi \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}(\cos 4\theta + 1) + 2\cos 2\theta + 1\right) d\theta \\
&= \frac{1}{3}\pi \left[\frac{1}{8}\sin 4\theta + \sin 2\theta + \frac{3}{2}\theta\right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{3}\pi \left(\frac{3}{2}\pi\right) = \frac{\pi^2}{2}.
\end{aligned}$$

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