MATH 23a, FALL 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 9
Due: December 12, 2003

1. Read the Appendix to Edwards (especially Theorems A.1 and A.6–A.8), Section 1.3 of Edwards, and Chapter 7 (especially sections 7.1–7.3) of Schneider and Barker.

2. (*) Prove that $\sqrt{5}$ is an irrational real number. (Hint: Think about equivalence classes in $\mathbb{Z}/5\mathbb{Z}$.)

3. (*) Prove that the sum of a rational number and an irrational real number must be an irrational real number.

4. (*) Let a and b be real numbers. Using P to denote the set of positive elements, we define "less than" formally by the statement:

$$a < b \quad \text{iff} \quad b - a \in P.$$

   Use axioms P1–P3 for an ordered field and the definition of absolute value to show that:

   - If $0 < a < b$, then $0 < a^2 < b^2$.
   - If $a^2 < b^2$, then $|a| < |b|$.

5. (*) Use the Completeness Axiom to prove that $\mathbb{N}$ is not bounded above as a set of real numbers.

6. (A) In this problem, we will show that the *golden ratio*, $\varphi = \frac{1+\sqrt{5}}{2}$, is a real number because it is the supremum of a non-empty bounded set of real numbers. (More precisely, we will show that φ is the limit of a bounded, increasing sequence.)

   Consider the recursively defined sequence:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \sqrt{1 + a_n}, \text{ for } n \geq 1$$

   (a) Use induction to show that $a_n \leq 2$ for all $n \in \mathbb{N}$.

   (b) Use induction to show that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

   (c) Since $L = \lim_{n\to\infty} a_n$ exists by the completeness axiom for $\mathbb{R}$, show that L satisfies the equation $L^2 - L - 1 = 0$ by considering the expression $\lim_{n\to\infty} a_{n+1}^2 - a_n - 1$.

(Hint: For this part, you may use theorems about limits such as the fact that $\lim_{n\to\infty}(a_n + b_n) = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} b_n\right)$ provided that these limits exist.)

(d) Show that $L = \varphi$.

7. (B) Recall homework problem #7.4. You showed that for $V = \mathbb{R}^n$ and $\mathbf{u}, \mathbf{v} \in V$, if $A : V \longrightarrow V$ is a linear transformation, then

$$f_A(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t A \mathbf{v}$$

defines a bilinear form.

Give a necessary and sufficient condition on A that makes f_A an inner product. (Full points for a complete answer in the $n = 2$ case.)

8. (C) Consider the real vector space $V = C[0, 1]$ of continuous real-valued functions defined on the closed interval $[0, 1]$, and define the bilinear form $\langle \cdot\, , \cdot \rangle : V^2 \to \mathbb{R}$ by:

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx$$

In class, we checked that this form is indeed bilinear, and it is clear that it is symmetric and positive. Use facts from single-variable Calculus to prove that this form is positive-definite.

(Hint: Look up the δ-ε definition of continuity!)

9. (D) Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$ and an associated norm $||\cdot||$. We say that a linear transformation $A : V \longrightarrow V$ is *norm-preserving* if $||A\mathbf{v}|| = ||\mathbf{v}||$ for every $\mathbf{v} \in V$ and *inner-product-preserving* if $\langle A\mathbf{u}, A\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$.

   (a) Show that A is inner-product-preserving if and only if it is norm preserving. (Hint #1: One way is easy! Hint #2: Expand the identity $||A(\mathbf{u} + \mathbf{v})||^2 = ||\mathbf{u} + \mathbf{v}||^2$.)

   (b) Let $V = \mathbb{R}^2$ with the usual inner product. Find all norm-preserving linear transformations/matrices.

10. (E) A vector space V is called a *normed linear space* if there exists a map $||\cdot|| : V \to \mathbb{R}$ called the **norm** satisfying:

   i. $||c \cdot \mathbf{v}|| = |c| \cdot ||\mathbf{v}||, \forall c \in \mathbb{R}, \forall \mathbf{v} \in V$

   ii. $||\mathbf{v}|| \geq 0, \forall \mathbf{v} \in V$, and $||\mathbf{v}|| = 0$ iff $\mathbf{v} = \mathbf{0}$.

   iii. $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||, \forall \mathbf{u}, \mathbf{v} \in V$

In Euclidean space, the map $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ is the norm *associated* to the inner product.

Show that the function $|| \cdot || : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$||(x_1, \ldots, x_n)|| = \max\{|x_1|, \ldots, |x_n|\}$$

defines a norm on $\mathbb{R}^n$. When $n > 1$, show that there is no inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ such that this norm is associated to the inner product.

11. (**) We will have covered this material by the end of classes, and you are responsible for it on the final, though it would be unfair to ask you to do it for homework before we have covered it in class.

Orthogonalizing a set of functions:

(a) Consider the vector space $V = C[-1, 1]$ of real-valued continuous functions on the closed interval $[-1, 1]$ with inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

Orthogonalize the set of functions $\{1, x, x^2, x^3\}$ with respect to this inner product.

(b) With $V = C[0, 1]$ and inner product

$$\langle f(x), g(x) \rangle = \int_{0}^{1} f(x)g(x)\, dx,$$

orthogonalize the same set of functions from part (a).

(c) Now let $V = C[-1, 1]$ as in part (a), but define $\langle \cdot, \cdot \rangle$ as in part (b). Show that this bilinear form is *not* an inner product on V. (Which properties of an inner product *are* satisfied?)