MATH 23A SOLUTION SET #9 (PART A)

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Problem (3). *Let V be a k-dimensional vector space over a field F. Show that a skew-symmetric multilinear form $f : V^n \to F$ is not necessarilly alternating.*

Solution. Most people realized that an example of a skew-symmetric but not alternating multilinear form f is easy to construct if we take $F = \mathbb{Z}/2\mathbb{Z}$, and giving an example when $n = 2$ and $k = 2$ was sufficient to get full credit.

So let $F = \mathbb{Z}/2\mathbb{Z}$, $V = F^2$. For any $v_1 = (a, b)$, $v_2 = (c, d) \in V$ define $f(v_1, v_2) = ac + bd$. Note that f is just the usual dot product on $V = (\mathbb{Z}/2\mathbb{Z})^2$, so it will not be difficult to show it is bilinear.

f **is bilinear.** Let $\alpha \in \mathbb{Z}/2\mathbb{Z}$. Then:

$$f(\alpha v_1, v_2) = f((\alpha a, \alpha b), (c, d)) = \alpha ac + \alpha bd = \alpha(ac + bd) = \alpha f(v_1, v_2)$$

and similarly $f(v_1, \alpha v_2) = \alpha f(v_1, v_2)$. Let $v_3 = (e, f) \in V$. Then:

$$f(v_1+v_3, v_2) = f((a+e, b+f), (c, d)) = (a+e)c+(b+f)d = (ac+bd)+(ec+fd) = f(v_1, v_2)+f(v_3, v_2)$$

and similarly, $f(v_1, v_2 + v_3) = f(v_1, v_2) + f(v_1, v_3)$.

f **is skew-symmetric.** Just recall that in $\mathbb{Z}/2\mathbb{Z}$, $1 = -1$, so that:

$$f(v_1, v_2) = ac + bd = ca + db = (-1)(ca + db) = (-1)f(v_2, v_1)$$

f **is not alternating.** To show f is not alternating, it is enough to exhibit a vector $v \in V$ such that $f(v, v) \neq 0$. So consider $v = (1, 0)$:

$$f(v, v) = f((1, 0), (1, 0)) = 1^2 + 0^2 = 1 \neq 0$$

□

Using the simple example above as a model, it is not difficult to come up with a general instance of a skew-symmetric non-alternating n-linear form on $V = (\mathbb{Z}/2\mathbb{Z})^k$. For $v_1, v_2, \ldots, v_n \in V$, with the notation $v_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,k})$, define

$$f(v_1, v_2, \ldots, v_n) = v_{1,1}v_{2,1} \ldots v_{n,1} + v_{1,2}v_{2,2} \ldots v_{n,2} + \ldots + v_{1,k}v_{2,k} \ldots v_{n,k}$$

Notice that our example above is a special case when $n = k = 2$. Proof that f is n-linear and skew-symmetric runs exactly as above, and we again have that, for $v = (1, 0, \ldots, 0)$,

$$f(v, v, \ldots, v) = 1^n + 0^n + \ldots + 0^n = 1 \neq 0$$

so that f is not alternating.