

# Solution Set 9C

Math 23a  
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5. This problem is done much more easily if we use the standard basis and write everything in matrix form, so we fix the standard basis  $(\mathbf{e}_1, \dots, \mathbf{e}_n)$  of  $\mathbf{R}^n$ . We know from problem seven that, given a basis, every matrix  $A$  gives rise to a bilinear form  $f_A$ ; we must show that every bilinear form arises in this way. Indeed, given a bilinear form  $f$ , define a matrix  $A_f$  by

$$A_f = \begin{bmatrix} f(\mathbf{e}_1, \mathbf{e}_1) & \cdots & f(\mathbf{e}_1, \mathbf{e}_n) \\ \vdots & \ddots & \vdots \\ f(\mathbf{e}_n, \mathbf{e}_1) & \cdots & f(\mathbf{e}_n, \mathbf{e}_n) \end{bmatrix}.$$

A simple calculation shows that  $f_{A_f}(\mathbf{e}_i, \mathbf{e}_j) = f(\mathbf{e}_i, \mathbf{e}_j)$ , so since  $f$  and  $f_{A_f}$  agree on a basis, they must be equal. Therefore  $f$  can indeed be expressed in matrix form.

More formally, if  $\mathcal{F}_n$  is the vector space of bilinear forms on  $\mathbf{R}^n$  and  $M_n(\mathbf{R})$  is the vector space of matrices with entries in  $\mathbf{R}$ , we define  $\varphi : M_n(\mathbf{R}) \rightarrow \mathcal{F}_n$  by  $\varphi(A) = f_A$ ; the above shows that this map is invertible, and it is easy to see that it is linear. Therefore the two vector spaces are isomorphic, so it suffices to work with  $\mathcal{F}_n$ .

If  $A_n(\mathbf{R}) \subset M_n(\mathbf{R})$  is the vector space of matrices that give rise to alternating forms then we know from problem seven that every  $A \in A_n(\mathbf{R})$  is skew-symmetric, i.e. has the property that  $A^t = -A$  (where  $A^t$  is the transpose of  $A$ ), and furthermore that every skew-symmetric matrix induces an alternating form. Since  $A_n(\mathbf{R})$  is isomorphic to the vector space of alternating bilinear maps on  $\mathbf{R}^n$  (by the above analysis), it suffices to work over this vector space.

- a) Since  $f((a, b), (a, b)) = ab - ba = 0$ ,  $f$  is alternating and therefore skew-symmetric. We also have

$$\begin{aligned} f((a, b) + (a', b'), (c, d)) &= (a + a')d - (b + b')c \\ &= (ad - bc) + (a'd - b'c) \\ &= f((a, b), (c, d)) + f((a', b'), (c, d)) \\ f(k(a, b), (c, d)) &= (ka)d - (kb)c \\ &= k(ad - bc) \\ &= kf((a, b), (c, d)) \end{aligned}$$

so  $f$  is linear in the first coordinate. Since  $f(\mathbf{v}, \mathbf{w}) = -f(\mathbf{w}, \mathbf{v})$ , we get linearity in the second coordinate for free; for instance,

$$f(\mathbf{v}, k\mathbf{w}) = -f(k\mathbf{w}, \mathbf{v}) = -kf(\mathbf{w}, \mathbf{v}) = kf(\mathbf{v}, \mathbf{w}).$$

Therefore  $f$  is bilinear and alternating.

Another way of seeing this is to note that

$$f((a, b), (c, d)) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

so  $f = f_A$ , where  $A$  is the matrix above. Note that  $A$  is skew-symmetric.

- b) Now the power of the matrix view becomes evident. We will implicitly identify matrices with the induced bilinear form from now on.

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Then  $A$  and  $B$  are visibly skew-symmetric and linearly independent, so they are linearly independent alternating bilinear forms.

- c) We know that

$$A_2(\mathbf{R}) = \left\{ \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} : a \in \mathbf{R} \right\}$$

which is clearly a one-dimensional vector space, with basis  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  (the form given in part (a)). Another way to show this is to note that we proved in class that the space of  $n$ -linear alternating forms on an  $n$ -dimensional vector space is one-dimensional; set  $n = 2$  to obtain  $\dim A_2(\mathbf{R}) = 1$ .

In a similar fashion,

$$A_3(\mathbf{R}) = \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} : a, b, c \in \mathbf{R} \right\}$$

so  $\dim A_3(\mathbf{R}) = 3$ . We can take a basis to be

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

which correspond to the forms

$$\begin{aligned} f_A((a, b, c), (a', b', c')) &= ab' - ba' \\ f_B((a, b, c), (a', b', c')) &= ac' - ca' \\ f_C((a, b, c), (a', b', c')) &= bc' - cb'. \end{aligned}$$

Notes on this problem:

- (1) Many people tried to argue along these lines: first, prove that any bilinear form is given as a product of pairs of coordinates of the first and second vectors, and second, figure out exactly which products can actually occur. There is nothing inherently wrong with this line of reasoning, although both steps are accomplished by simply expanding a bilinear form out in terms of a basis. However, many people had faulty arguments for the first step; most people started by assuming that if  $f$  is a bilinear form then

$$f((x_1, x_2, x_3), (x'_1, x'_2, x'_3)) = \sum c_i \cdot (\text{products involving } x_i \text{ and } x'_i).$$

This is true, but one cannot assume it; one has to prove it. Simply saying “if something is bilinear then it has to look like a linear combination of products” is not a proof. The way to prove such a fact is to actually expand out using the axioms of bilinearity, but once you’ve done that, you’re done anyway.

- (2) In part (a), maybe only one in five people actually proved all of the bilinearity axioms. There are four of them:

$$f(\mathbf{v} + \mathbf{v}', \mathbf{w}) = f(\mathbf{v}, \mathbf{w}) + f(\mathbf{v}', \mathbf{w})$$

$$f(\mathbf{v}, \mathbf{w} + \mathbf{w}') = f(\mathbf{v}, \mathbf{w}) + f(\mathbf{v}, \mathbf{w}')$$

$$f(k\mathbf{v}, \mathbf{w}) = kf(\mathbf{v}, \mathbf{w})$$

$$f(\mathbf{v}, k\mathbf{w}) = kf(\mathbf{v}, \mathbf{w}).$$

The reason that these are all stated separately is that in general, they are *independent* axioms, that is, no combination of them implies another. In other words, if you proved linearity in the first coordinate, that does not imply linearity in the second coordinate (unless you proved that it is alternating *first*, in which case you should mention that alternating-ness implies linearity in the second coordinate). The proof of linearity in the second coordinate is, of course, very similar, but you have to say that — you can’t just forget about it.

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