

MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm Solutions (in-class portion)

1. **Properties of Number Systems**

For each entry in the following table, identify whether the number system at the head of the column has that property or not by placing a “Y” for yes and an “N” for no.

Property	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{Z}/6\mathbb{Z}$
Existence of Additive Inverses	N	Y	Y	Y	Y
Existence of Multiplicative Inverses	N	N	Y	Y	N
Ordered	Y	Y	Y	Y	N
Well-ordered	Y	N	N	N	N
Complete	X	Y	N	Y	X

2. **True or False**

- (a) **T or F** If $L : V \longrightarrow V$ is an injective linear map, then it is surjective.

False. We have seen examples (such as the shift operator on the space of infinite sequences) that are injective without being surjective. (The statement *is* true if $\dim(V) < \infty$.)

- (b) **T or F** If $L : V \longrightarrow W$ is a bijective linear map, then $V \cong W$.

True. This is the definition of isomorphic.

- (c) **T or F** If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset V$ is a set of linearly independent vectors, then any $\mathbf{v} \in V$ may be written as a linear combination of these vectors in a unique way.

False. If a vector could be written as such a linear combination, the representation would be unique, but the set of vectors might not span V .

- (d) **T or F** If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset V$ is a set of vectors such that none of them is a scalar multiple of any of the others, then the set is linearly independent.

False. It is possible to have a non-trivial linear combination giving zero even if no two are scalar multiples of each other.

- (e) **T or F** If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset V$ is a set of vectors such that $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} = V$, then $\dim(V) \leq n$.

True. This is a theorem from class.

- (f) **T or F** If $L : V \rightarrow V$ is a linear transformation, then $\text{Ker}(L)$ is a subspace of V .

True. This is a theorem from class.

- (g) **T or F** The set of vectors $\{(0, 1, 2), (1, 1, 1), (0, 0, 0)\}$ is linearly independent in $\mathbb{R}^3$.

False. No set of vectors including $\mathbf{0}$ can be linearly independent. In this example, we have $0 \cdot (0, 1, 2) + 0 \cdot (1, 1, 1) + 1 \cdot (0, 0, 0) = (0, 0, 0)$, even though not all the coefficients are 0.

- (h) **T or F** If $V = \text{span}\{(0, 1, 2, 3, 4), (1, 1, 1, 1, 1)\}$, then $\dim(V) = 5$.

False. Since the two vectors are not scalar multiples of each other, they are linearly independent, and $\dim(V) = 2$.

3. Let $\{a_n\}$ and $\{b_n\}$ be sequences of rational numbers.

(a) Define what it means for $\{a_n\}$ to be a Cauchy sequence.

The sequence $\{a_n\}$ is Cauchy means that, given any $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ whenever $n, m > N$.

(b) If $\{a_n\}$ and $\{b_n\}$ are both Cauchy sequences, show that $\{a_n \cdot b_n\}$ (the term-by-term product of the original two sequences) is also a Cauchy sequence.

You may either cite the following lemma from class or prove it directly as follows:

Lemma Any Cauchy sequence $\{a_n\}$ is bounded. That is, there exists some $M \in \mathbb{R}$ such that $|a_n| < M$, for every $n \in \mathbb{N}$.

Proof: Since the sequence is Cauchy, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ whenever $n, m > N$. This implies that for every $n > N$,

$$|a_n - a_{N+1}| < 1$$

and hence

$$-1 < a_n - a_{N+1} < 1$$

or

$$a_{N+1} - 1 < a_n < a_{N+1} + 1$$

which implies that $|a_n| < \max\{|a_{N+1} - 1|, |a_{N+1} + 1|\}$ for every $n > N$. Finally, let

$$M = \max\{|a_1|, |a_2|, \dots, |a_N|, |a_{N+1} - 1|, |a_{N+1} + 1|\}.$$

This M bounds the sequence.

Now, we move on to the proof of the statement. Let M_1 and M_2 be the bounds for the sequences $\{a_n\}$ and $\{b_n\}$, respectively, as given in the lemma.

Given $\varepsilon > 0$, choose $N = \max\{N_1, N_2\}$, where N_1 is such that $|a_n - a_m| < \frac{\varepsilon}{2M_2}$, for all $n, m > N_1$, and N_2 is such that $|a_n - a_m| < \frac{\varepsilon}{2M_1}$, for all $n, m > N_2$. Then:

$$\begin{aligned} |a_n \cdot b_n - a_m \cdot b_m| &= |(a_n \cdot b_n - a_m \cdot b_n) + (a_m \cdot b_n - a_m \cdot b_m)| \\ &= |b_n||a_n - a_m| + |a_m||b_n - b_m| \\ &= M_2 \cdot \frac{\varepsilon}{2M_2} + M_1 \cdot \frac{\varepsilon}{2M_1} \\ &= \varepsilon, \end{aligned}$$

where the inequality follows from the fact that $n, m > N$.

4. Let $T : V \rightarrow V$ be a non-zero linear map. We consider the situation when $T^2 = 0$.

(Note that $T^2 = T \circ T$, in other words, T composed with itself, and 0 is the zero linear map that takes every vector in V to 0 .)

- (a) Show that $T^2 = 0$ if and only if $\text{Im}(T) \subset \text{Ker}(T)$.

($\Rightarrow$) Assume $T^2 = 0$.

Suppose $\mathbf{v} \in \text{Im}(T)$. Then there is some $\mathbf{w} \in V$ such that $T(\mathbf{w}) = \mathbf{v}$. By the assumption, we see that $T(\mathbf{v}) = T(T(\mathbf{w})) = T^2(\mathbf{w}) = 0$, and hence $\mathbf{v} \in \text{Ker}(T)$.

($\Leftarrow$) Assume $\text{Im}(T) \subset \text{Ker}(T)$.

Consider any $\mathbf{v} \in V$. We need to show that $T^2(\mathbf{v}) = 0$. Note that $T(\mathbf{v}) \in \text{Im}(T)$ and that therefore $T(\mathbf{v}) \in \text{Ker}(T)$, by assumption. This implies that $T(T(\mathbf{v})) = 0$, which is what we needed.

- (b) Show that if $T^2 = 0$ and $\dim(V) = 3$, then $\dim(\text{Ker}(T)) = 2$.

By the Rank-Nullity Theorem, we know that $\dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = \dim(V) = 3$, and by the result from part (a), since $\text{Im}(T) \subset \text{Ker}(T)$, it follows that $\dim(\text{Im}(T)) \leq \dim(\text{Ker}(T))$. Putting these together, we see that $2 \cdot \dim(\text{Im}(T)) \leq 3$, which means that $\dim(\text{Im}(T)) = 0$ or 1 , but since T is supposed to be non-trivial, in fact, $\dim(\text{Im}(T)) = 1$, and hence $\dim(\text{Ker}(T)) = 2$.

- (c) Give an example of a non-trivial vector space V and a non-trivial linear map $T : V \rightarrow V$ satisfying $T^2 = 0$.

Three examples:

1. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $T(x, y) = (0, x)$
2. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $T(x, y) = (x - y, x - y)$
3. $D : P_1 \rightarrow P_1$, which is the usual differentiation operator such that $D(ax + b) = a$.

5. Let $P_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ be the vector space of polynomials of degree three or less with real coefficients. Consider the linear map $I : P_3 \rightarrow \mathbb{R}$ defined by $I(p(x)) = \int_0^1 p(x) dx$, for $p(x) \in P_3$.

(a) Find $\dim(\text{Ker}(I))$.

By the Rank-Nullity Theorem, we know that

$$\dim(\text{Im}(I)) + \dim(\text{Ker}(I)) = \dim(P_3) = 4.$$

Since $\dim(\text{Im}(I)) \subset \mathbb{R}$, we see that $\dim(\text{Im}(I)) \leq 1$. Since $I(2cx) = \int_0^1 2cx dx = cx^2 \Big|_0^1 = c$, we see that any $c \in \mathbb{R}$ is also in $\text{Im}(I)$, and hence $\text{Im}(I) = \mathbb{R}$, or $\dim(\text{Im}(I)) = 1$.

Thus, $\dim(\text{Ker}(I)) = 3$.

(b) Exhibit a basis for $\text{Ker}(I)$.

Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in \text{Ker}(I)$. Then

$$\begin{aligned} I(p(x)) &= \int_0^1 (a_0 + a_1x + a_2x^2 + a_3x^3) dx \\ &= a_0x + \frac{1}{2}a_1x^2 + \frac{1}{3}a_2x^3 + \frac{1}{4}a_3x^4 \Big|_0^1 \\ &= a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 \\ &= 0 \end{aligned}$$

Choosing $a_0 = 1$ and $a_1 = -2$ yields one non-zero polynomial $p_1(x) = 1 - 2x$.

Choosing $a_0 = 1$ and $a_2 = -3$ yields another $p_1(x) = 1 - 3x^2$.

Finally, choosing $a_0 = 1$ and $a_3 = -4$ yields $p_1(x) = 1 - 4x^3$.

(c) Prove that your answer to part (b) is in fact a basis.

We check (using the degrees of these polynomials) that these three vectors are linearly independent. Suppose $c_1 \cdot p_1(x) + c_2 \cdot p_2(x) + c_3 \cdot p_3(x) = 0$. Then $(c_1 + c_2 + c_3) + (-2c_1)x + (-3c_2)x^2 + (-4c_3)x^3$ is the zero polynomial, but the only way this can be true is if $c_1 = c_2 = c_3 = 0$.

Since any three linearly independent vectors form a basis, we are done.

6. Let V be a vector space over the field F , and let W be a subspace of V . We define the quotient space V/W as follows:

$$V/W = \{\mathbf{v} + W \mid \mathbf{v} \in V\} / \sim,$$

where $\mathbf{v}_1 + W \sim \mathbf{v}_2 + W$ if and only if $\mathbf{v}_1 - \mathbf{v}_2 \in W$.

The elements of the quotient space are called cosets, and they have the form

$$\mathbf{v} + W = \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in W\}.$$

If we define addition and scalar multiplication as follows:

$$(\mathbf{v}_1 + W) + (\mathbf{v}_2 + W) = (\mathbf{v}_1 + \mathbf{v}_2) + W$$

$$c \cdot (\mathbf{v} + W) = (c \cdot \mathbf{v}) + W$$

for any $c \in F$ and any $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V$, then in fact, the quotient space V/W is a vector space over F .

- (a) It is a fact that $\sim$ is an equivalence relation. Show that $\sim$ is *transitive*.

Suppose $\mathbf{v}_1 + W \sim \mathbf{v}_2 + W$ and $\mathbf{v}_2 + W \sim \mathbf{v}_3 + W$. Then by the definition of equivalence, these imply that $\mathbf{v}_1 - \mathbf{v}_2 \in W$ and $\mathbf{v}_2 - \mathbf{v}_3 \in W$, respectively. Since W is a subspace, it is closed under addition, and hence $\mathbf{v}_1 - \mathbf{v}_3 = (\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{v}_2 - \mathbf{v}_3) \in W$. Again by the definition of equivalence, this implies that $\mathbf{v}_1 + W \sim \mathbf{v}_3 + W$.

- (b) It is a fact that addition and scalar multiplication are well-defined. Show that *addition* is well-defined.

Suppose $\mathbf{v}_1 + W \sim \mathbf{v}_2 + W$ and $\mathbf{u}_1 + W \sim \mathbf{u}_2 + W$. Then by the definition of equivalence, these imply that $\mathbf{v}_1 - \mathbf{v}_2 \in W$ and $\mathbf{u}_1 - \mathbf{u}_2 \in W$, respectively. Again, since W is a subspace, it is closed under addition, and hence $(\mathbf{v}_1 + \mathbf{u}_1) - (\mathbf{v}_2 + \mathbf{u}_2) = (\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{u}_1 - \mathbf{u}_2) \in W$. By the definition of equivalence, this implies that $(\mathbf{v}_1 + \mathbf{u}_1) + W \sim (\mathbf{v}_2 + \mathbf{u}_2) + W$, and hence addition is well-defined.

- (c) It is a fact that V/W is a vector space over F . Show that V/W satisfies axiom V3 concerning *additive identities*.

It is easy to check that $\mathbf{0} + W$ is the additive identity since $(\mathbf{0} + W) + (\mathbf{v} + W) = (\mathbf{0} + \mathbf{v}) + W = \mathbf{v} + W$, and $(\mathbf{v} + W) + (\mathbf{0} + W) = (\mathbf{v} + \mathbf{0}) + W = \mathbf{v} + W$ as well.