Last Name: ______________________________

First Name: ______________________________

MATH 23a, FALL 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm (in-class portion)
October 31, 2003

Directions: You have one hour (more like 53 minutes) for this exam. No calculators, notes, books, etc. are allowed. Please answer on the pages provided. Show all work!

Problem	Points	per part	Score
1	40	2 each	
2	8		
3	16	4/4/8	
4	16	4 each	
5	15	5 each	
Total	95	95	

1. **True or False**

T or **F** $\mathbb{Z}/4\mathbb{Z}$ is a field.

T or **F** $\mathbb{Z}/5\mathbb{Z}$ is a commutative ring with identity.

T or **F** In a vector space, additive inverses are unique.

T or **F** The empty set is a vector space over any field.

T or **F** Every non-trivial vector space contains infinitely many vectors.

T or **F** Every vector space (except the trivial vector space) has a basis.

T or **F** Any three non-zero vectors span $\mathbb{R}^3$.

T or **F** The vector space $\mathbb{R}^3$ has a basis containing the vector $(1, 2, 3)$.

T or **F** For p a prime, the vector space $V = (\mathbb{Z}/p\mathbb{Z})^2$ has $(p^2 - 1)(p^2 - p)$ (ordered) bases.

T or **F** If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $dim(V) = n$.

T or **F** If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a set of vectors such that none of them is a scalar multiple of any of the others, then the set is linearly independent.

T or **F** If $V \cong W$, then there is a bijective linear map $L : V \longrightarrow W$.

T or **F** Isomorphism of vector spaces is an equivalence relation.

T or **F** If V is a subspace of U and $dim(V) < \infty$, then $dim(U/V) < \infty$.

For the remainder of the True/False questions, let $L : U \to V$ be a linear map.

T or **F** If $Ker(L) = \{\mathbf{0}\}$, then L is injective.

T or **F** If $dim(U) = dim(V)$, then L is bijective.

T or **F** $Ker(L)$ is a subspace of U.

T or **F** If $dim(U) < \infty$, then $dim(Im(L)) < \infty$.

T or **F** If $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for U, then $\{L(\mathbf{u}_1), \ldots, L(\mathbf{u}_n)\}$ is a basis for V.

T or **F** If $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a linearly independent set, then $\{L(\mathbf{u}_1), \ldots, L(\mathbf{u}_n)\}$ is also a linearly independent set.

2. State the Principle of Mathematical Induction.

3. Let $L : U \to V$ be a linear map between two finite-dimensional vector spaces, and suppose that $dim(U) > dim(V)$.

    (a) Define (in terms of the elements of U and V) what it would mean for L to be *surjective*.

    (b) Multiple choice (circle one):

        i. L must be surjective.

        ii. L could be, but is not necessarily, surjective.

        iii. L cannot be surjective.

    (c) Use the Rank-Nullity Theorem to prove that L cannot be injective.

4. For any $n \in \mathbb{N}$, recall that we define

$$P_n(\mathbb{R}) = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}, \forall i\}$$

to be the vector space of polynomials of degree less than or equal to n with real coefficients.

(a) Construct a basis for $P_3(\mathbb{R})$ containing $p(x) = 1 - x + x^2 - x^3$ as one of its vectors.
(You don't need to show it to be a basis in this part.)

(b) Show that your basis from part (a) does in fact span $P_3(\mathbb{R})$.

(c) Give an example of a non-zero linear map $L : P_3(\mathbb{R}) \to P_2(\mathbb{R})$.

(d) Show that your map L from part (c) is in fact linear.

5. Let $\ell = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \mathbb{R}, \forall i \in \mathbb{N}\}$ be the vector space of all sequences of real numbers, and consider the shift operator $S : \ell \to \ell$ given by:

$$S(a_1, a_2, a_3, a_4, \ldots) = (a_2, a_3, a_4, \ldots).$$

Let S^2 and S^3 be the composition of S with itself 2 and 3 times, respectively, and let $T = S^3 - S^2 - S$.

(a) For a vector $(a_1, a_2, a_3, \ldots) \in \ell$, write out $T(a_1, a_2, a_3, \ldots)$ explicitly, including at least the first four terms of the resulting sequence.

(b) Find a basis for $Ker(T)$.
(You don't need to show it to be a basis in this part.)

(c) Show that your basis from part (b) is in fact linearly independent.