Problem Set 2, Part C – Solutions

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(d) Consider the vector $(0,1)$ and all its scalar multiples. Then the following set is a non-trivial subspace of V:

$$U = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6)\}$$

One can easily check that it is closed under addition and scalar multiplication, because all vectors in this set are of the form $a(0,1)$ for some $a \in \{0, 1, ..., 6\}$.

So, let v_1 and v_2 be two vectors in this subspace. Then:

- $v_1 + v_2 = a(0,1) + b(0,1) = (a+b)(0,1)$, which is in V
- $a \cdot v = ab(0,1)$ which is also in V, for all $a \in \mathbb{Z}/p\mathbb{Z}$ and all $v\ inV$.

So, U is closed under addition and scalar multiplication $\Rightarrow U$ is a subspace of V.

(e) Now, to count the subspaces of V, we use the same general idea. First, we want to see how many vectors are in V. Since there are 7 possible numbers in $\mathbb{Z}/p\mathbb{Z}$, then the number of vectors in V is $7^2 = 49$. Since the vector $(0,0)$ is contained in all subspaces of V, we are left with 48 vectors.

Now consider an arbitrary non-zero vector $v \in V$. Assume that two of its scalar multiples are equal. This means that there exist $a, b \in \mathbb{Z}/p\mathbb{Z}$ such that $av = bv \Rightarrow (a-b)v = 0$. But since 7 is a prime number, this equation is possible if and only if $v = 0$ or $a - b = 0$. Since v was chosen to be different than 0, then we must have $a - b = 0 \Rightarrow a = b$.

Moreover, non-zero scalar multiples of vectors which are linearly independent , are different. This is fairly easy to check. Assume $av_1 = bv_2 \Rightarrow av_1 - bv_2 = 0 \Rightarrow v_1$ and v_2 are linearly dependent − contradiction.

So all scalar multiples of a given vector are distinct and scalar multiples of linearly independent vectors are also distinct. Moreover, as we've seen in part (d), a vector together with its scalar multiples forms a subspace of V. Since there are 6 possible scalar multiples of a given vector (including itself but excluding (0,0)), then we obtain $48/6 = 8$ non-trivial subspaces.

Assume there exist another subspace U, which contains at least two different vectors, say v_1, v_2 such that none of them is a scalar multiple of the others. Then, U must contain all scalar multiples of these two vectors. Also, it has to contain the sum $v_1 + v_2 = v_3$.

Obviously, v_3 cannot be a scalar multiple of v_1 or v_2. Assume it is $\Rightarrow$ one of v_1 or v_2 is a scalar multiple of the other. Contradiction with our assumptions.

So U must also contain all scalar multiples of v_3, which are all different from the scalar multiples of v_1 and v_2. If one keeps repeating this procedure, one will eventually realize that all vectors in V will have to be in $U \Rightarrow U = V$.

In conclusion, there couldn't be any other non-trivial subspaces in addition to the 8 that we obtained above.

To these 8, we add two which are trivial, that is $\{0\}$ and $V \Rightarrow$ the total number of subspaces of is 10.