10. Let V be a vector space over the field F defined as all infinite sequences of elements of F:

$$V = \{(a_0, a_1, a_2, \ldots) \mid a_i \in F, \forall i \geq 0\}$$

For $k = 1, 2$, we define the two subspaces:

$$l^k = \{(a_0, a_1, a_2, \ldots) \mid a_i \in F, \forall i \geq 0, \text{ and } \sum_{n=0}^{\infty} |a_n|^k \text{ converges } \}$$

b. Let $F = \mathbf{R}$. Show that l^2 is a subspace of V by showing that the addition of vectors and scalar multiplication are closed operations. That is, if $u, v \in l^2$, then $u + v \in l^2$ and $au \in l^2$.

We'll start with scalar multiplication because this is much easier. Let $v \in l^2$ s.t. $v = (v_0, v_1, \ldots)$. Then we know that $\sum_{i=0}^{\infty} |v_i|^2$ converges. For any $a \in \mathbf{R}$, $av = (av_0, av_1, \ldots)$. We need to show that $\sum_{i=0}^{\infty} |av_i|^2$ converges. But we know that $\sum_{i=0}^{\infty} |av_i|^2 = \sum_{i=0}^{\infty} a^2|v_i|^2 = a^2 \sum_{i=0}^{\infty} |v_i|^2$, which is clearly convergent. So $av \in l^2$ and we are done.

Now let $u, v \in l^2$. We hope to show that $u + v \in l^2$. We start by determining precisely what this means. Because $u, v \in l^2$, we have that $\sum_{i=0}^{\infty} |u_i|^2$ and $\sum_{i=0}^{\infty} |v_i|^2$ converge. We now investigate $\sum_{i=0}^{\infty} |u_i + v_i|^2$. Using simple algebra we know that

$$\sum_{i=0}^{\infty} |u_i + v_i|^2 \leq \sum_{i=0}^{\infty} |u_i|^2 + 2|u_i v_i| + |v_i|^2 = \sum_{i=0}^{\infty} |u_i|^2 + \sum_{i=0}^{\infty} 2|u_i v_i| + \sum_{i=0}^{\infty} |v_i|^2$$

By our assumption the outside series converge, so we see that $u + v \in l^2$ iff $\sum_{i=0}^{\infty} 2|u_i v_i|$ converges. We know that for each i, $2|u_i v_i|$ is less than or equal to either $2u_i^2$ or $2v_i^2$. However, it is entirely possible that for some i and j $u_i > v_i$ but $v_j > u_j$ so that $\sum_{i=0}^{\infty} 2|u_i v_i|$ is not necessarily less than one of $\sum_{i=0}^{\infty} 2|u_i|^2$ or $\sum_{i=0}^{\infty} 2|v_i|^2$. However, we can conclude that $2|u_i v_i| \leq 2u_i^2 + 2v_i^2$ for each i. So the sum $\sum_{i=0}^{\infty} 2|u_i v_i|$ is bounded by

$$\sum_{i=0}^{\infty} 2|u_i|^2 + 2|v_i|^2 = 2\sum_{i=0}^{\infty} |u_i|^2 + 2\sum_{i=0}^{\infty} |v_i|^2$$

which converges. So $\sum_{i=0}^{\infty} 2|u_i v_i|$ converges as well and $u + v \in l^2$.

Note: Many of you attempted to use the ratio test to show that $u + v \in l^2$ by manipulating the limits of the terms in this sequence. Aside from several difficulties that arise in doing arithmetic with limits, the ratio test, if used correctly, cannot be completely conclusive in this case. The reason is this: the ratio test says that if the limit of the ratio of successive terms in the series is less than 1, the series converges; if the ratio is greater than 1, the series diverges; AND if the ratio equals 1, we cannot tell if the series will converge or diverge. So if we have a sequence $(a_0, a_1, \ldots) \in l^2$, we know that $lim_{n\to\infty} \frac{a_{n+1}^2}{a_n^2} \leq 1$, not that this limit is less than 1 (because our series could be in the equals one case). Conversely, if you show that the limit is less than or equal to one, we CANNOT conclude that the series converges, again because of the equals one case.

c. Let $F = \mathbf{R}$. Show that $l^1 \subset l^2$. Show by example that $l^1 \neq l^2$.

Let's start with a word on notation. The phrase $l^1 \subset l^2$ means show that l^1 as a set is contained in the set l^2. If you read this to mean, show that l^1 is a subspace of l^2 you would still need to show that it is a subset before you should even think about checking the axioms of a subspace. So we'll show that l^1 is a subset of l^2 here.

To show that $l^1 \subset l^2$, we must argue that for all sequences $v \in l^1$, we also have $v \in l^2$. As before, the ratio test won't work here because we might have a limit equal to (and not less than) 1. Also, we can't just square the series $\sum_{i=0}^{\infty} |v_i|$ because we don't have a clear idea of what it means to multiply two series (even though we know exactly what this would be if the series were finite). But we do have a fair amount of information about the terms in the series, so that is probably the best way to start.

We claim that because $\sum_{i=0}^{\infty} |v_i|$ converges, there exist only finitely many indices i such that $|v_i| \geq 1$. This is clearly true because if there were infinitely many $|v_i| \geq 1$ then our sum would be infinite. Furthermore, for all i such that $|v_i| < 1$, $|v_i|^2 < |v_i|$. So we can consider two subseries of our series $\sum_{i=0}^{\infty} |v_i|$, the one with $|v_i| \geq 1$ and the one with $|v_i| < 1$. Notice, I'm not saying that we will reorder our series from largest to smallest because this cannot always be done (why?). Instead let $I = \{i : |v_i| \geq 1\}$ which we have seen is a finite set. Then $\sum_{i=0}^{\infty} |v_i| = \sum_{i \in I} |v_i| + \sum_{i \notin I} |v_i|$. Similarly, we write $\sum_{i=0}^{\infty} |v_i|^2 = \sum_{i \in I} |v_i|^2 + \sum_{i \notin I} |v_i|^2$ and note that these index sets I are the same (why?). We now can state what we determined above: that $\sum_{i \notin I} |v_i|^2$ converges because it is bounded by $\sum_{i=0}^{\infty} |v_i|$. So what about $\sum_{i \in I} |v_i|^2$? It's true that these terms are likely much larger than those of the series $\sum_{i \in I} |v_i|$. However, it's important to note that we are summing finitely many terms. Thus, no matter how large the $|v_i|^2$ are, the sum is finite (i.e., convergent). So we conclude that $\sum_{i=0}^{\infty} |v_i|^2$ converges and $v \in l^2$.

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is an example of a non-convergent series that will converge when the terms are squared. So the harmonic sequence (i.e. $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$) is a sequence that is not in l^1 but is in l^2. Thus $l^2 \neq l^1$.

d. Let $F = \mathbf{Z}/2\mathbf{Z}$. Note that by "converges", we mean (as always!) that the sequence of partial sums has a limit.

Given an explicit condition for deciding whether or not a particular sequence is in l^1 or l^2. (What must happen for such an infinite series to converge? Recall that the limits and sums are being taken in the finite field!)

We note that $[1]^2 = [1]$ and $[0]^2 = [0]$ so our answer will be the same for both l^1 and l^2. In either case, we will be adding an infinite string of [0]s and [1]s. As long as we keep running into [1]s along the sequence, the partial sums will oscillate back and forth between [0] and [1]. So if this is the case, the sum will not converge. That means that we must have a limit to the number of [1]s that appear in our sequence. More precisely, the sum will converge iff the number of [1]s that appear is finite. Put another way, there must exist some n such that for all $N > n$, $a_N = [0]$.