2. Let V be a vector space, and let $\{v_1, \ldots, v_n\} \subset V$ be a collection of linearly independent vectors. Show that no collection of $n-1$ vectors spans V.

There are actually two cases to consider in this problem: $\dim V = \infty$ and $\dim V < \infty$. If V is infinite dimensional, then our problem is easy, because no finite set may span an infinite dimensional vector space by definition. So now we can assume that V is a finite dimensional vector space (note: I did not take off any points for failing to mention the infinite dimensional case, because no one remembered to do so, but you should keep things like this in mind in the future).

There are a number of correct ways to prove that no collection of $n-1$ vectors spans V, the simplest of which involve a proof by contradiction. Thus, we let $\{w_1, \ldots, w_{n-1}\} \subset V$ be a set of vectors that spans V. I will present two of the most popular solutions here.

Solution 1. *Because $\{w_1, \ldots, w_{n-1}\}$ spans V, there exists some subset $\{w'_1, \ldots, w'_k\}$ of these vectors that is a basis for V with $k \leq n-1$. At this point we can apply a theorem proven in class. The theorem states that given a linearly independent spanning set of vectors of some space W, no collection of vectors $\{v_1, \ldots, v_n\} \subset W$ with $n > k$ can be linearly independent. It is important to note that the reason we can use this theorem here is because $\{w'_1, \ldots, w'_k\}$ spans V and not some subspace of V that might not include all of the v_i. So, as $k \leq n-1 < n$ the vectors $\{v_1, \ldots, v_n\}$ must be linearly dependent, a contradiction. So $\{w_1, \ldots, w_{n-1}\}$ cannot span V.*

Note: you can only apply this theorem with a ***basis*** *because it requires that the spanning set of vectors are linearly independent. It is very important to carefully check all of the conditions in a theorem before you apply it to a specific problem, which is why I took off so many points for those of you who overlooked this. Take it as a warning and be more careful in the future.*

Solution 2. *The second solution is similar. As before we reduce our spanning set $\{w_1, \ldots, w_{n-1}\}$ to a subset $\{w'_1, \ldots, w'_k\}$ of these vectors that is a basis for V with $k \leq n-1$. Thus we conclude that the dimension of V is equal to k. But we have another result that says that any linearly independent set may be extended to a basis of a finite dimensional vector space. So we may extend the set $\{v_1, \ldots, v_n\}$ to a set $\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_m\}$ of linearly independent vectors that spans V, another basis. So we conclude that $\dim V = m$ with $m \geq n$. But we must have $m = n$ which is clearly impossible as $m \geq n > n-1 \geq k$. Thus we have a contradiction and $\{w_1, \ldots, w_{n-1}\}$ cannot span V.*