
Math 23a Solution Set #3, Part E

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Problem 8

Let U and W be subspaces of a vector space V . We define two new subspaces as follows:

$$\begin{aligned}U + W &= \{u + w \mid u \in U, w \in W\} \\U \cap W &= \{v \in V \mid v \in U \text{ and } v \in W\}.\end{aligned}$$

- (a) (*) Convince yourself that both $U + W$ and $U \cap W$ are, in fact, subspaces.

Solution: This is just a matter of showing that $U + W$ and $U \cap W$ are closed under addition and scalar multiplication.

- (b) Show that if $\dim(V) < \infty$, then

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Solution: Since $\dim(V) < \infty$, we know that the dimension of any subspace of V is finite. Let (v_1, \dots, v_n) be a basis of $U + W$. Because it is linearly independent, we can extend (v_1, \dots, v_n) to bases $\mathcal{B}(U) = (v_1, \dots, v_n, u_1, \dots, u_p)$ of U and $\mathcal{B}(W) = (v_1, \dots, v_n, w_1, \dots, w_q)$ of W . If we can show that

$$\mathcal{B} = (v_1, \dots, v_n, u_1, \dots, u_p, w_1, \dots, w_q)$$

is a basis of $U + W$, then we're done, because in that case

$$\begin{aligned}\dim(U + W) &= n + p + q \\&= (n + p) + (n + q) - n \\&= \dim(U) + \dim(W) - \dim(U \cap W).\end{aligned}$$

To finish the proof, we must show that \mathcal{B} is a linearly independent set that spans $U + W$.

If $v \in U + W$, then $v = u + w$ for some $u \in U, w \in W$. Since $\mathcal{B}(U)$ is a basis of U and $\mathcal{B}(W)$ is a basis of W , we can write $u = a_1v_1 + \dots + a_nv_n + b_1u_1 + \dots + b_pu_p$ and $w = c_1v_1 + \dots + c_nv_n + d_1w_1 + \dots + d_qw_q$ for some scalars a_i, b_i, c_i, d_i . But then $v = u + w = (a_1 + c_1)v_1 + \dots + (a_n + c_n)v_n + b_1u_1 + \dots + b_pu_p + d_1w_1 + \dots + d_qw_q$. Hence \mathcal{B} spans $U + W$.

Finally, we show that \mathcal{B} is linearly independent. Suppose that

$$a_1v_1 + \cdots + a_nv_n + b_1u_1 + \cdots + b_pu_p + d_1w_1 + \cdots + d_qw_q = 0 \quad (1)$$

for some scalars a_i, b_i, d_i . Consider the vector $w = d_1w_1 + \cdots + d_qw_q$. Clearly $w \in W$. But $w \in U$ as well, because we can write it as the linear combination $w = (-a_1)v_1 + \cdots + (-a_n)v_n + (-b_1)u_1 + \cdots + (-b_p)u_p$. Hence $w \in U \cap W$, and we can write $w = c_1v_1 + \cdots + c_nv_n = d_1w_1 + \cdots + d_qw_q$ for some scalars c_i . Since $\mathcal{B}(W)$ is linearly independent, it follows that all the d_i 's (and c_i 's) are zero. Now equation (1) becomes

$$a_1v_1 + \cdots + a_nv_n + b_1u_1 + \cdots + b_pu_p = 0.$$

Since $\mathcal{B}(U)$ is linearly independent, all of the a_i 's and b_i 's are zero. We have shown that the a_i 's, b_i 's and d_i 's are zero, hence \mathcal{B} is linearly independent.

Remarks

- Many people had a lot of trouble doing this problem. The idea is to extend a basis of $U \cap W$ to bases of U and W , and take the union of the bases of U and W to obtain a basis of $U + W$. You must show that the basis you construct is indeed a basis, that is, it is a linearly independent set of vectors that spans $U + W$.
- About half of you structured the problem correctly as above, but did not prove that their basis of $U + W$ was linearly independent. (The other half of you did not structure the problem correctly). Many people said something like “the basis of $U + W$ contains the vectors in basis of $U \cap W$ twice, so we must subtract $\dim(U \cap W)$ from $\dim(U) + \dim(W)$ ”. This might make sense intuitively, but it is not convincing enough. You have to show that every vector in $U + W$ has a unique representation as a linear combination of its basis vectors. In other words, if a linear combination of basis vectors is zero, then each coefficient in that linear combination must be zero.
- Some people began the problem by choosing arbitrary bases for U and W . But these bases are not helpful for solving the problem. Note that there need not be a subset of vectors in the bases of U and W which constitute a basis of $U \cap W$; the bases of U and W may be disjoint. If you are going to be talking about the dimension of $U \cap W$, then you know something about a basis of $U \cap W$.
- A depressing number of people began the problem by saying “let U' be the subspace $U - (U \cap W)$ and W' be the subspace $W - (U \cap W)$...”. This is nonsense. U' and W' don't contain the zero vector, so they are not vector spaces, and it does not make sense to talk about the “dimension” of one of these spaces.
- A couple of people used a proof given in a textbook. (For instance, see Edwards, p. 33). This is fine, but make sure you cite your work and understand what you're doing. One person copied the proof from Edwards almost word for word, without realizing that the proof refers to an earlier result.
- **Please keep your answers concise!** I spend about 10-20 seconds looking at each problem set. If you write a lot then the chances are I will be too lazy to read it and give you a lower score.