Math 23a Solution Set #4, Part A

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Problem 3

Let $P_n(\mathbb{R}) = \{p(x) = a_0 + a_1x + \cdots + a_nx^n \mid a_i \in \mathbb{R}, \forall i\}$ be the vector space of all polynomials of degree less than or equal to n. Consider the map $L : P_n(\mathbb{R}) \to \mathbb{R}$ defined by $L(p) = \int_0^1 p(x)dx$.

(a) *Show that L is a linear map.*

Solution: This just follows from the linearity of integration. For all $p_1, p_2 \in P_n(\mathbb{R})$, we have

$$L(p_1 + p_2) = \int_0^1 p_1(x) + p_2(x)dx = \int_0^1 p_1(x)dx + \int_0^1 p_2(x)dx = L(p_1) + L(p_2),$$

and for all $p \in P_n(\mathbb{R})$ and $c \in \mathbb{R}$, we have

$$L(cp) = \int_0^1 cp(x)dx = c\int_0^1 p(x)dx = cL(p).$$

(b) *Determine* $\text{Im}(L)$*, and find a basis.*

Solution: Note that $1 = \int_0^1 1dx \in \text{Im}(L)$. Since $\text{Im}(L)$ is a subspace of $\mathbb{R}$, we know that $c \cdot 1 \in \text{Im}(L)$ for all $c \in \mathbb{R}$. Hence $\text{Im}(L) = \mathbb{R}$ and $\{1\}$ is a basis.

Problem 4

In this problem, we consider the shift *operator. Consider the linear map $S : \mathbb{R}^3 \to \mathbb{R}^3$ which acts as follows:*

$$S(x, y, z) = (0, x, y).$$

Find the kernel and image of S, and verify that

$$\dim(\ker(S)) + \dim(\text{Im}(S)) = \dim(\mathbb{R}^3).$$

Solution: The kernel of S is the space $\{(0, 0, z) \mid z \in \mathbb{R}\}$ and the image is $\{(0, x, y) \mid x, y \in \mathbb{R}\}$. The first has basis $\{(0, 0, 1)\}$ and the second has basis $\{(0, 1, 0), (0, 0, 1)\}$. Hence $\dim(\ker(S)) = 1$ and $\dim(\text{Im}(S) = 2$, and the problem reduces to showing that $1 + 2 = 3$.