5. We generalize the notion of the shift operator. Let V be the vector space of all infinite sequences of real numbers as in problem 3.9, and consider the linear maps $S : V \to V$ and $T : V \to V$, where S and T act as follows:

$$S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$$

$$T(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$$

a. Find the kernel and image of S. Does the result about the dimensions of kernels and images (the Rank-Nullity Theorem) apply?

The kernel of S is the set of vectors $v \in V$ such that $S(v) = 0$. By the definition of S, this requires that $a_i = 0$ for $i = 0, 1, \ldots$, so we may conclude that $v = 0$. So $\ker(V) = \{0\}$. Similarly, the image of S is the set of all vectors $w \in V$ such that $w = S(v)$ for some $v \in V$. So in this case, $im(V) = \{(0, a_0, a_1, a_2, \ldots) | a_i \in F\}$.

The Rank-Nullity Theorem does not apply to this linear map because V is infinite dimensional (consider the "standard basis" $\{v_0, v_1, v_2, \ldots\}$ where v_i has $a_i = 1$ and all other $a_j = 0$). Clearly these vectors are linearly independent and span V so we can say $\dim V = \infty$.

b. Show that $T \circ S = I$ but that $S \circ T \neq I$, where $I : V \to V$ is the identity map.

By definition $I(v) = v$ for all $v \in V$. So we show that $T \circ S(v) = v$. We compute

$$T \circ S(v) = T(S(v)) = T(S(a_0, a_1, \ldots)) = T(0, a_0, a_1, \ldots) = (a_0, a_1, \ldots) = v$$

So $T \circ S = I$. The other composition, we find isn't true. Let $v \in V$ such that $v = (1, 0, 0, \ldots)$. Then $S \circ T(v) = S(T(v)) = S(0, 0, \ldots) = (0, 0, 0, \ldots) \neq v$. So $S \circ T \neq I$.

c. Which of S and T is onto? Which is one-to-one? Which is invertible? Explain.

From part a we know that S is injective, as $\ker(S) = \{0\}$. However, we see that T is not injective, because $(1, 0, 0, \ldots) \in \ker(T)$ and this element is non-zero (remember 0 is always in the kernel of a linear map).

To show that T is onto we consider a general vector $v = (a_0, a_1, a_2, \ldots) \in V$ and find a vector $v' \in V$ such that $Tv' = v$. We see that if we let $v' = (1, a_0, a_1, a_2, \ldots)$ then $v' \in V$ and $Tv' = v$. Thus V is onto. However S is not onto. The vector $(1, 0, 0, \ldots)$ is in V but not in the image of S.

We conclude that neither T nor S are invertible as both maps fail to be bijective. Note that it is irrelevant that $T \circ S = I$ because $S \circ T \neq I$ (though for finite dimensional vector spaces one part will imply the other).