Solution for HW4, part D
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Problem 7

Let $A : U \to V$ and $B : V \to W$ be linear maps between finite dimensional vector spaces.
(a) Show that if A and B are both surjective, then so is $B \circ A$.

(b) Show that if B is one-to-one, then $Ker(B \circ A) = Ker(A)$.

(c) Assuming A and B are surjective, show that $dim(Ker(B \circ A)) = dim(Ker(A)) + dim(Ker(B))$.

Part a

Here is one proof. First a bit of notation. Let $M : S \to T$ be some map (not necessarily linear). If $S' \subset S$, define $M(S') = \{M(s)|s \in S'\}$. So this is just notation for the image of a set under a map. In particular, $im(M) = M(S)$. Notice that M is surjective iff $M(S) = T$. Let's use this notation to do the proof:

We need to show $(B \circ A)(U) = W$. You can check that $(B \circ A)(U)$ is just $B(A(U))$. Surjectivity of A implies $A(U) = V$, so $B(A(U)) = B(V) = W$, since B is surjective as well.

This proof does not depend on the fact that A and B are linear. It is a general fact of maps that the composition of surjective maps is again surjective.

Part b

A bit more notation: let $M : S \to T$. If $T' \subset T$, define $M^{-1}(T') = \{s \in S|M(s) \in T'\}$. Or, if T' consists of a single element, t, we write $M^{-1}(t) = \{s \in S|M(s) = t\}$. Keep in mind that this does *not* assume that M is invertible. This is a way of naming the set of elements that map to a particular set. Notice that in the case of a linear map, $Ker(M) = M^{-1}(0)$. Ok, so now the proof:

let's do things a bit more generally so that we can see what's going on. I claim $Ker(B \circ A) = A^{-1}(KerB)$. That is, $Ker(B \circ A)$ is the set of things that map via A to the kernel of B. We can check this: $a \in Ker(B \circ A) \Leftrightarrow B(A(a)) = 0 \Leftrightarrow A(a) \in Ker(B) \Leftrightarrow a \in A^{-1}(Ker(B))$ establishes the equality of the two sets. In this problem, we have that B is injective, i.e. $Ker(B) = \{0\}$. Hence $Ker(B \circ A) = A^{-1}(0) = Ker(A)$.

Part c

Time for some use of rank-nullity. A few people mistakenly thought that injectivity of

B carried over from the previous problem, but Prof. Boller wanted the proof in the more general setting. Onward!

Surjectivity of A and B imply surjectivity of $B \circ A$, by part a, so we have $im(A) = V$, $im(B) = W$ and $im(B \circ A) = W$. Using these facts, along with rank-nullity, we get:

$dim(Ker(A)) = dim(U) - dim(V)$,

$dim(Ker(B)) = dim(V) - dim(W)$ and

$dim(Ker(B \circ A)) = dim(U) - dim(W)$. The result of of the problem follows directly.

Another angle of attack would be to construct a basis. The basic idea is this: let $(v_1, \ldots, v_n)$ be a basis for $Ker(B)$. Since A is surjective, we can take preimages $(u_1, \ldots, u_n)$ such that $A(u_i) = v_i$, $i = 1 \ldots n$. Now let $(u_{n+1}, ldots, u_{n+m})$ be a basis for $Ker(A)$. Then $(u_1, \ldots, u_{n+m})$ is a basis for $Ker(B \circ A)$. You can check this is a basis in the usual way, by checking span and linear independence.