Problem Set 4, Part E – Solutions

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Solution

Define the following map $L : C[0,1] \to C[a,b]$ such that: $(L(f))(x) = f(\frac{x-a}{b-a})$. Note that we can divide by $b-a$ since $a \neq b$.

Consider the output of L as a composition of two functions, $f \circ g$ where $g : [a,b] \to [0,1]$ with $g(x) = \frac{x-a}{b-a}$. The range of this function is well-defined since $g(a) = 0$ and $g(b) = 1$ and for all $x \in (a,b), 0 < g(x) < 1$.

Now it is clear why $f \circ g : [a,b] \to [0,1]$. Since both f and g are continuous functions, their composition is also a continuous function.

To check whether our map is bijective, we find its inverse L^{-1} in the following way:

$$L^{-1} \circ L = \text{identity} \Rightarrow L^{-1}(L(f)) = f \Rightarrow (L^{-1}(f)) \left(\frac{x-a}{b-a} \right) = f(x) \tag{1}$$

Denote $y = \frac{x-a}{b-a}$ with $y \in [0,1]$ when $x \in [a,b] \Rightarrow y(b-a) = x-a \Rightarrow x = y(b-a)+a$.

So now, equation 1 becomes: $(L^{-1}(f))(y) = f(y(b-a)+a)$ with $L^{-1} : C[a,b] \to C[0,1]$ so this means that we found the inverse.

Note that it is very easy to check from this point that $L \circ L^{-1} = \text{identity}$.

So, the only thing left to prove now is that L is linear.

- given f_1, f_2 two functions in $C[a,b]$, then $L(f_1+f_2)(x) = (f_1+f_2)(\frac{x-a}{b-a}) = f_1(\frac{x-a}{b-a}) + f_2(\frac{x-a}{b-a}) = L(f_1)(x) + L(f_2)(x)$
- given $f \in C[a,b]$ and $k \in R$, then $L(k \cdot f)(x) = (k \cdot f)(\frac{x-a}{b-a}) = k \cdot f(\frac{x-a}{b-a})$

Since these two conditions are satisfied, then L is linear.

In conclusion, we exhibited a linear bijection between two vector spaces $\Rightarrow$ they are isomorphic.